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ON THE TRANSFORMATIONS OF THE LOGARITHMIC SERIES

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In this paper we consider transformations of the series

$$l(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \text{ and } L(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1}$$

in the forms: (A) $l(x) = \sum_{n=1}^{\infty} \frac{A_n x^n}{1-\alpha_n x}$, (B) $L(z) = \sum_{n=0}^{\infty} \frac{B_n}{1-b_n z^2} \left(\frac{z}{1-\beta_n z^2}\right)^{4n+1}$ and (C) $l(x) = \sum_{n=1}^{\infty} \frac{C_n x^n}{(1-\gamma_1 x)\cdots(1-\gamma_n x)}$. Minimization of the coefficients in (A) and (B), under the restrictions $|\alpha_n|, |\beta_n| \le 1$, is explored numerically. The resulting hypothesis is that we can accelerate the convergence like a geometric progression. We prove that the unique lacunary series $l(x) = \sum_{i=0}^{\infty} \frac{A_i x^{2i+1}}{1-\alpha_i x}$ and $L(z) = \sum_{i=0}^{\infty} \frac{B_i z^{4i+1}}{1-b_i z^2}$ diverge for $x \neq 0$ and $z \neq 0$. Assuming $|\gamma_n| \le 1$ we prove lower and upper bounds for the optimal rate of convergence of (C). A similar upper bound for (A) is proved. Also, some new accelerated series for the logarithmic and other transcendental functions are obtained.

Keywords: Logarithm, Series acceleration, Recurrences, Rational approximation.

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1. INTRODUCTION

In this study we consider some rational transformations of the series

$$f(x) := a_1 x + a_2 x^2 + a_3 x^3 + \cdots , \tag{1}$$

which is assumed to have radius of convergence equal to 1. Mainly, we restrict our attention to representations of $l(x) := \ln\Big(\dfrac{1}{1-x}\Big)$, i.e. with $a_n = \dfrac{1}{n}$, and other related functions of the forms

$$f(x) \approx \frac{A_1 x}{1-\alpha_1 x} + \frac{A_2 x^2}{1-\alpha_2 x} + \frac{A_3 x^3}{1-\alpha_3 x} + \cdots \tag{2}$$

and

$$f(x) \approx \frac{C_1 x}{1-\gamma_1 x} + \frac{C_2 x^2}{(1-\gamma_1 x)(1-\gamma_2 x)} + \frac{C_3 x^3}{(1-\gamma_1 x)(1-\gamma_2 x)(1-\gamma_3 x)} + \cdots . \tag{3}$$

The symbol "$\approx$" can be considered as coincidence of formal power series, or as asymptotic expansion for $x \to 0$. The goal is to obtain series that converge faster than the initial one and that coincide with the corresponding function in a neighborhood of $x = 0$. The form (2) is a sum of geometric series, while (3) is similar to a Newton series and having the same computational efficiency as (2) it allows much easier treatment.

Everywhere in this paper, if the area of validity of an equality involving series is not specified, then it can be considered as certain neighborhood of the origin or more specifically, the disk $\{w \in \mathbb{C} : |w| < |p_1|\}$, where p_1 is the closest to 0 non-zero singular point (sometimes 0 will be a removable singularity).

As there are extremely fast methods for computing the logarithmic function (see e.g. [4, Ch.1.3]), transformations (2) and (3) of (1) do not bring something new in this area. Actually, $l(x)$ serves as a model function in studying the possibilities of the forms like (2) and (3) for acceleration of power series. Such transformations can occur in calculating other transcendental functions like $Li_k(x)$ or the Euler digamma function. Another aim of the study is to point out to some interesting and difficult analytical problems which appear meanwhile.

Note that for the transformation of (1) in the form (2) (similarly for (3)) the convergence of the series does not matter. Given $\{a_n\}$, if we fix the series $\{\alpha_n\}$, then the numbers $\{A_n\}$ in (2) are obtained easily by the recursive formulas

$$A_1\alpha_1^{n-1} + A_2\alpha_2^{n-2} + \cdots + A_{n-1}\alpha_{n-1} + A_n = a_n. \tag{4}$$

Conversely, if we choose in advance $\{A_n\}$, then the numbers $\{\alpha_n\}$ are obtained by the same formulas, provided no division by zero is encountered. Formally, it is an easy task to rewrite the series (1) in the form (2) with coefficients $\{A_n\}$ that tend arbitrarily fast to 0. However, the requirement the series in (2) to converge to $f(x)$ in a neighborhood of $x = 0$ poses the restriction on the poles $\{1/\alpha_n\}$ to be distinct from zero, that is, the sequence of parameters $\{\alpha_n\}$ to be bounded.

What we have is a coding of the power series (1) by using twice as much parameters $\{A_n, \alpha_n\}$ (or $\{C_n, \gamma_n\}$). From this point of view we arrive at an extremal problem of optimizing over the extra parameters according to certain minimization criterion. We shall try to formulate simple criteria in order to decompose the minimization of the overall series $\{A_n\}$ ($\{C_n\}$) by greedy type algorithms, which determine the series step by step. Also, we study numerically other rational forms generalizing (2).

Finally, lacunary series are of great interest. We shall prove that the unique transformation of $l(x)$ in the type (2) with $A_{2k} = 0, k = 1, 2, \dots$ is divergent. In contrast, it is easy to obtain lacunary representations of $l(x)$ in the type (3) that converge. Actually, the well known series for $\ln\Big(\dfrac{1+z}{1-z}\Big)$, $|z| < 1$, can be written as

$$l(x) = 2\Big(z + \frac{z^3}{3} + \frac{z^5}{5} + \frac{z^7}{7} + \cdots\Big) =: 2L(z), \quad z = \frac{x}{2-x}, \tag{5}$$

which is of type (3) with parameters sequences $\{\gamma_n\} = \left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots\right\}$ and $\{C_n\} = \left\{\frac{1}{1}, 0, \frac{1}{3}\Big(\frac{1}{2}\Big)^2, 0, \frac{1}{5}\Big(\frac{1}{2}\Big)^4, 0, \dots,\right\}$. Also, this example shows that there is a choice of a bounded sequence $\{\gamma_n\}$ in (3) for $l(x)$, having rate of convergence of $\{C_n\}$ as a geometric series with ratio $\frac{1}{2}$.

The paper is organized as follows. In Section 2 some classical methods for accelerating series are applied to $l(x)$ and $L(z)$. In Section 3 we describe numerical experiments for optimization of the representations of $l(x)$ and $zL(z)$ in the form (2) and the lacunary form $f(x) \approx \sum_i \frac{B_i}{1-b_i x} \frac{x^{2i+1}}{(1-\beta_i x)^{k_i}}$. Using different algorithms we found parameter sequences such that $|\alpha_i|(|\beta_{i-1}|) \le 1$ and $|A_i|(|B_{i-1}|) \le q^{i-1}$, $i = 1, \dots, i_1$ $(q < 1)$. The above representation with $\beta_i = p$ is of particular interest. This special case is partially investigated for convergence in Section 4. As a result, the following theorem is proved there:

Theorem 1. *The unique lacunary representations*

$$a)\ l(x) \approx \sum_{i=0}^{\infty} \frac{A_i x^{2i+1}}{1 - a_i x} \quad \text{and} \quad b)\ L(z) \approx \sum_{i=0}^{\infty} \frac{B_i z^{4i+1}}{1 - b_i z^2}$$

are divergent for every nonzero value of the argument.

In Section 5 we consider the representation (3) for $l(x)$ and prove the following

Theorem 2. *Let $\{C_n\}$ and $\{\gamma_n\}$ be the parameters in (3) for $f(x) = l(x)$. Then, for every $\varepsilon \in (0, 1]$,*

a) there exists a choice of $\{\gamma_n\}$ such that $\gamma_n \in [0, 1]$ and the corresponding coefficients satisfy $|C_n| < M(4-\varepsilon)^{-n}$ for every $n \in \mathbb{N}$ with some $M = M(\varepsilon)$.

b) there is no choice of $\{\gamma_n\}$ such that $\gamma_n \in [0, 1]$ and $|C_n| < M(8+\varepsilon)^{-n}$ for every $n \in \mathbb{N}$ with some $M = M(\varepsilon)$.

As a consequence of this we obtain

Theorem 3. *Let $f(x) = l(x)$ and the parameters $\{\alpha_n\}_1^\infty$, $\{\gamma_n\}_1^\infty$ satisfy the restrictions $|\alpha_n|, |\gamma_n| \le 1$. Then for the sequences $\{A_n\}_1^\infty$ and $\{C_n\}_1^\infty$ determined by (2) and (3) correspondingly, there is no positive number M such that*

$$|A_n| \le M \cdot 31^{-n} \ \text{for every} \ n \in \mathbb{N} \quad \text{or} \quad |C_n| \le M \cdot 25^{-n} \ \text{for every} \ n \in \mathbb{N}.$$

Also, in this section some concrete series with periodic $\{\gamma_n\}$ are obtained and a comparison of the series (3) with continued fraction representation for $l(x)$ is done. Finally, in Section 6 we consider some accelerated series for other transcendental functions, including $Li_2(x)$ and $\psi(x)$.

We finish the introductory section with presenting another point of view. The form (2) can be considered as a power series with varying coefficients, i.e., $f(x) \approx \sum_{n=0}^{\infty} F_n(x).x^n$, where $\{F_n(x)\}$ are functions of a specific class (in (2), $F_n(x) = \frac{A_n}{1-\alpha_n x}$). Obviously, the simplest choice $F_n(x) = A_n + B_n x$ brings nothing for the acceleration of (1). The next natural choice actually is the complete linear fractional transformation $F_n(x) = \frac{A_n+B_n x}{C_n+D_n x}$. This form perhaps deserves more attention than (2) because of the following property, which is preserved by the form (3), but not by (2). Namely, if the first n poles $\{\gamma_i^{-1}\}_{i=1}^{n}$ in (3) interchange their order, then the residual (and the n-th partial sum) do not change. Similarly, in the above generalization of (2), we can change the order of two poles, with an appropriate change of the other parameters, so that the residual of the series remains the same. Indeed, let $S = \dfrac{\alpha_n + \beta_n x}{1-\gamma_n x}x^n + \dfrac{\alpha_{n+1} + \beta_{n+1} x}{1-\gamma_{n+1} x}x^{n+1}$ be the sum of two consecutive terms. Then, if $\gamma_{n+1} \neq 0$, we have the identity $S = \dfrac{\bar{\alpha}_n + \bar{\beta}_n x}{1-\gamma_{n+1} x}x^n + \dfrac{\bar{\alpha}_{n+1} + \bar{\beta}_{n+1} x}{1-\gamma_n x}x^{n+1}$, where $\bar{\alpha}_n = \alpha_n$, $\bar{\beta}_n = \alpha_{n+1} - \alpha_n\gamma_{n+1} + \frac{\beta_{n+1}}{\gamma_{n+1}}$, $\bar{\alpha}_{n+1} = \alpha_n\gamma_n + \beta_n - \frac{\beta_{n+1}}{\gamma_{n+1}}$ and $\bar{\beta}_{n+1} = \gamma_n\frac{\beta_{n+1}}{\gamma_{n+1}}$. In the exceptional case $\gamma_{n+1} = 0$ we have $S = \dfrac{\alpha_n + \alpha_{n+1} x}{1-0.x}x^n + \dfrac{\alpha_n\gamma_n + \beta_n}{1-\gamma_n x}x^{n+1} + \beta_{n+1}x^{n+2}$ and the last summand can be joined to the next term in the series.

2. SOME SIMPLE EXAMPLES

Let us consider the case $\alpha_n = 1, n = 1, 2, 3, \ldots$. Then it is easily verified that

$$l(x) = \frac{1}{1-x}\Big(x - \frac{x^2}{1.2} - \frac{x^3}{2.3} - \frac{x^4}{3.4} - \cdots\Big).$$

This is a Kummer type acceleration but also it can be explained as follows. $l(x)$ has a singularity at $x = 1$ which have logaritmic order divergence. Then $(1-x)l(x)$ is “more regular”, having at least finite limit when $x \to 1$. This explains why the later function has smaller Maclaurin series than $l(x)$. Following this line of reasoning, for every $r \in \mathbb{N}$, we can write the acceleration formula:

$$\frac{1}{r!}\Big(1-\frac{1}{x}\Big)^r l(x) = P_{r-1}\Big(\frac{1}{x}\Big) + \sum_{n=1}^{\infty} \frac{x^n}{n(n+1)\ldots(n+r)},$$

where $P_{r-1}(z)$ is a polynomial of degree $r-1$. The proof easily follows if we substitute in the infinite sum $\frac{1}{n(n+1)\ldots(n+r)} = \frac{(-1)^r}{r!}\Delta^r\frac{1}{n}$ by $\frac{1}{r!}\sum_{k=0}^{r}\binom{r}{k}\frac{(-1)^k}{n+k}$. For example, when $r=2$ it follows that ([3, 1.513])

$$\Big(1-\frac{1}{x}\Big)^2 l(x)=\frac{1}{x}-\frac{3}{2}+2\sum_{n=1}^{\infty}\frac{x^n}{n(n+1)(n+2)}.$$

Since it is not easy to improve formula (5) for $l(x)$, by the end of this section we are going to accelerate $L(z)$. Similarly as above we get

$$\Big(\frac{1}{z}-z\Big)L(z)=1-2z\Big(\frac{z}{1.3}+\frac{z^3}{3.5}+\frac{z^5}{5.7}+\frac{z^7}{7.9}+\cdots\Big)$$

and

$$\Big(\frac{1}{z}-z\Big)^2 L(z)=\frac{1}{z}-\frac{5}{3}z+8\Big(\frac{z^3}{1.3.5}+\frac{z^5}{3.5.7}+\frac{z^7}{5.7.9}+\cdots\Big).$$

For another type acceleration let us consider the changes of the variables

$$L(z)=z\sum_{n=0}^{\infty}\frac{z^{2n}}{2n+1}=z\sum_{n=0}^{\infty}\frac{t^n}{2n+1}=z\sum_{n=0}^{\infty}C_n\Big(\frac{t}{1-pt}\Big)^n=:zf(\tau),$$

where $t=z^2$, $\tau=\dfrac{t}{1-pt}$ and p is a real parameter. We shall see that the best choice for p, when the sequence $\{C_n\}$ decreases in the fastest way, is $p=\frac{1}{2}$. Indeed, since the change $\tau=\frac{t}{1-pt}$ and its inverse $t=\frac{\tau}{1+p\tau}$ are regular in a neighborhood of the origin, the same is true for the function $f(\tau)$. The radius of convergence of $f(\tau)$ depends on its smallest singular point. For real τ we have

$$f(\tau)=\begin{cases}\frac{1}{2\sqrt{t}}\ln\frac{1+\sqrt{t}}{1-\sqrt{t}}, & \text{for } t\in(0,1)\\ \frac{1}{2\sqrt{-t}}\arctan\sqrt{-t}, & \text{for } t\in[-1,0)\ ,\\ 1, & \text{for } t=0\end{cases}\qquad t=\frac{\tau}{1+p\tau}.$$

It is quite clear from this expression that the singular points of any analytic continuation of $f(\tau)$ are $\tau=-\frac{1}{p}$ and $\tau=\frac{1}{1-p}$, when $t=1$. (Note that $\tau=0$ is a removable singular point.) Then the radius of convergence of f is $R(p)=\min\{\frac{1}{|p|},\frac{1}{|1-p|}\}$ and it is easy to verify that $\max_{p\in\mathbb{R}}R(p)=R(1/2)=2$. As a result we conclude that the optimal acceleration of $L(z)$ by this transformation gives coefficients $\{C_n\}$ that tend to 0 like a geometric series with ratio $\frac{1}{2}$. Next, with $p=\frac{1}{2}$, it is easy to check out the identity $f(\tau)+\tau(2+\tau)f'(\tau)=\frac{1+\tau/2}{1-\tau/2}$ from where we find the recurrence formula

$$(2n+1)C_n+(n-1)C_{n-1}=2^{1-n},\quad n=1,2,3,\ldots\quad (C_0=1).$$

Thus, the transformed series starts as follows

$$L(z)=z\left[1+\frac{1}{3}\tau+\frac{1}{2.5!!}\tau^2+\frac{11}{4.7!!}\tau^3+\frac{39}{8.9!!}\tau^4+\frac{633}{16.11!!}\tau^5+\cdots\right],\quad \tau=\frac{z^2}{1-z^2/2}.$$

An acceleration of the same order but with more explicit coefficients can be obtained using Euler transform applied in certain succession. If $F(x) = \sum_{n=0}^{\infty} a_n x^n$ then the Euler transform is defined by the identity $\dfrac{1}{1+t}F\Big(\dfrac{t}{1+t}\Big) = \displaystyle\sum_{n=0}^{\infty}(\Delta^n a_0)t^n$, where $\Delta a_i = a_{i+1} - a_i$ and $\Delta^n a_i = \Delta(\Delta^{n-1}a_i)$. Sometimes by Euler transform it is understood the particular case for $x = -1$, i.e. when $t = -\frac{1}{2}$, which converts an alternating numerical series usually into a faster converging one. For $a_n = \frac{1}{2n+1}$, $n = 0, 1, 2, \dots$ it is easy to find that $\Delta^n a_0 = (-1)^n \frac{(2n)!!}{(2n+1)!!}$. Then, the Euler transform leads to

$$L(z) = \frac{z}{1-z^2}\sum_{n=0}^{\infty}\frac{(2n)!!}{(2n+1)!!}(-y)^n, \quad y = \frac{z^2}{1-z^2}, \tag{6}$$

which is the well known series ([3, 1.515])

$$\frac{\ln(\sqrt{y}+\sqrt{1+y})}{\sqrt{1+y}} = \sum_{n=0}^{\infty}(-1)^n\frac{(2n)!!}{(2n+1)!!}(\sqrt{y})^{2n+1}.$$

Note that the series (6) has approximately the same rate of convergence as (5), and if we apply the Euler transform to (6), then we return exactly at (5). Actually, the idempotence is a general property of the Euler transform after the change $y = -t$ (see [5]). The key observation for accelerating $L(z)$ in this way is that an application of the Euler transform from a larger index is more effective. So, leaving the first term in (5) unchanged and applying Euler transform to the residual we get

$$L(z) = z\left\{1 + \frac{z^2}{1-z^2}\Big[\frac{1}{3} - \frac{2!!}{5!!}y + \frac{4!!}{7!!}y^2 - \frac{6!!}{9!!}y^3 + - \cdots\Big]\right\}, \quad y = \frac{z^2}{1-z^2}.$$

Again leaving the first term in the square brackets and applying the Euler transform to the residual (with argument $-y$) we obtain

$$L(z) = z\left\{1 + \frac{1}{3}y - z^2 y\Big[\frac{2}{3.5} + \frac{2}{5.7}z^2 + \frac{2}{7.9}z^4 + \frac{2}{9.11}z^6 + \cdots\Big]\right\}.$$

Continuing in the same way we find

$$\begin{aligned}
L(z) &= z\left\{1 + \frac{1}{3}y - \frac{2}{3.5}yz^2 - 3y^2z^2\Big[\frac{2!!}{7!!} - \frac{4!!}{9!!}y + \frac{6!!}{11!!}y^2 - \cdots\Big]\right\}\\
&= z\left\{1 + \frac{1}{3}y - \frac{2!}{5!!}yz^2 - \frac{3!}{7!!}y^2z^2 + y^2z^4\Big[\frac{4!!}{5.7.9} + \frac{4!!}{7.9.11}z^2 + \frac{4!!}{9.11.13}z^4 + \cdots\Big]\right\}
\end{aligned}$$

$$= z\left\{1 + \frac{1}{3}y - \frac{2!}{5!!}yz^2 - \frac{3!}{7!!}y^2z^2 + \frac{4!}{9!!}y^2z^4 + 5!!y^3z^4\Big[\frac{4!!}{11!!} - \frac{6!!}{13!!}y + \frac{8!!}{15!!}y^2 - \cdots\Big]\right\}$$

and so on to arrive at the series

$$L(z) = z\left\{1 + \frac{1!}{3!!}y - \frac{2!}{5!!}yz^2 - \frac{3!}{7!!}y^2z^2 + \frac{4!}{9!!}y^2z^4 + \frac{5!}{11!!}y^3z^4 - \frac{6!}{13!!}y^3z^6 - \cdots\right\}. \tag{7}$$

The same result can be obtained more directly. Namely, starting from (6) and using the identities $\frac{1}{1-z^2} = 1 + y$ and $y^2 = yz^2(1+y)$ we can transform $L(z)$ as follows:

$$\begin{aligned}
L(z) &= z\left\{1+\frac{1}{3}y - y^2\Big[\frac{2!!}{5!!} - \frac{4!!}{7!!}y + \frac{6!!}{9!!}y^2 - \frac{8!!}{11!!}y^3 + \cdots\Big]\right\} \\
&= z\left\{1+\frac{1}{3}y - yz^2\Big[\frac{2!}{5!!} + \frac{3!}{7!!}y - \frac{3.4!!}{9!!}y^2 + \frac{3.6!!}{11!!}y^3 - \frac{3.8!!}{13!!}y^4 + - \cdots\Big]\right\} \\
&= z\left\{1+\frac{1}{3}y - \frac{2!}{5!!}yz^2 - \frac{3!}{7!!}y^2z^2 + 3y^3z^2\Big[\frac{4!!}{9!!} - \frac{6!!}{11!!}y + \frac{8!!}{13!!}y^2 - + \cdots\Big]\right\}
\end{aligned}$$

$$=z\left\{1+\frac{1}{3}y-\frac{2!}{5!!}yz^2-\frac{3!}{7!!}y^2z^2+3y^2z^4\Big[\frac{4!!}{9!!}+\frac{5.4!!}{11!!}y-\frac{5.6!!}{13!!}y^2+\frac{5.8!!}{15!!}y^3-\cdots\Big]\right\}$$

$$=z\left\{1+\frac{1}{3}y-\frac{2!}{5!!}yz^2-\frac{3!}{7!!}y^2z^2+\frac{4!}{9!!}y^2z^4+\frac{5!}{11!!}y^3z^4-5!!y^4z^4\Big[\frac{6!!}{13!!}-\frac{8!!}{15!!}y+\cdots\Big]\right\}$$

$$=z\left\{1+\frac{1}{3}y-\frac{2!}{5!!}yz^2-\frac{3!}{7!!}y^2z^2+\frac{4!}{9!!}y^2z^4+\frac{5!}{11!!}y^3z^4-5!!y^3z^6\Big[\frac{6!!}{13!!}+\frac{7.6!!}{15!!}y-\cdots\Big]\right\}$$

and so on. Finally, let us remark that formula (7) is of type (3) with $z^2 = x$.

3. SOME COMPUTER EXPERIMENTS

1. We start with the choice of the parameters $\{\alpha_n\}_{n=1}^{\infty}$ in the form (2) of $f(x) = l(x)$, suggested by the simplest greedy algorithm. Namely, we choose every next α_n such that $|A_{n+1}|$ to be minimal. Thus we arrive at a lacunary representation of $l(x)$. Let us explain the derivation of the first four coefficients. Clearly $A_1 = 1$ and the requirement $A_2 = 0$ leads, by (4), to the equation $A_1\alpha_1 + 0 = \frac{1}{2}$, i.e. to $\alpha_1 = \frac{1}{2}$. As a result of $A_2 = 0$, we have no control on A_3 and the relation $A_1\alpha_1^2 + A_2\alpha_2 + A_3 = \frac{1}{3}$ gives $A_3 = \frac{1}{12}$. Next, the choice $A_4 = 0$ is possible because the equation $A_1\alpha_1^3 + 0 + A_3\alpha_3 + 0 = \frac{1}{4}$ has a solution $\alpha_3 = \frac{3}{2}$. Continuing in this way we obtain

$\{\alpha_n\} = \{\frac{1}{2}, *, \frac{3}{2}, *, \frac{35}{12}, *, \frac{35077}{6324}, *, \frac{167344077283}{15930229780} = 10.504..., *, 19.899..., *, \ldots\}$;

$\{A_n\} = \{1, 0, \frac{1}{12}, 0, -\frac{1}{20}, 0, \frac{527}{4032}, 0, -\frac{1511407}{1214208}, 0, 42.385..., 0, -5174.4..., \ldots\}$,

where "*" means an arbitrary number. It is seen that the obtained series diverge rapidly and we shall prove this in the next section. An heuristic explanation is from the type of the recurrence relations (4). Once an $|\alpha_n|$ larger than 1 occurs, then larger and larger numbers will appear in (4), which most likely will draw $\{|A_n|\}$ to infinity. A similar behavior is observed in the following lacunary representation

$$L(z) \approx \frac{z}{1-\frac{1}{3}z^2} + \frac{\frac{4}{45}z^5}{1-\frac{25}{21}z^2} - \frac{\frac{4}{147}z^9}{1-\frac{1609}{693}z^2} + \frac{0.043699...\,z^{13}}{1-4.4448...\,z^2} - \frac{0.26698...\,z^{17}}{1-8.4284...\,z^2}\cdots.$$

2. In view of the above observations, in the next two examples we pose the requirement for the summands to be regular in the open unit disk, i.e. $|\alpha_{\mathbf{n}}| \leq \mathbf{1}$. We have that

$$l(x) \approx \frac{x}{1-0.2\,x} + \frac{0.3\,x^2}{1-0.8\,x} + \frac{x^3/18.75}{1-\ x} - \frac{x^4/300}{1+0.4\,x} - \frac{x^5/101.35...}{1-0.8\,x} - \frac{x^6/694.44...}{1-\ x} - \frac{x^7/767.54...}{1-0.x} - \frac{x^8/2425.6...}{1+\ x} - \frac{x^9/9582.6...}{1-\alpha_9\,x} \pm \cdots,$$

where the coefficients satisfy $|A_n| \leq 3^{1-n}$, $n = 1, ..., 9$. For the method used for obtaining this series see the next example. Now we formulate the following

Hypothesis 1. There is a choice of $\{\alpha_n\} \subset \mathbb{R}$ with $|\alpha_n| \leq 1$ such that the coefficients in the form (2) of $l(x)$ satisfy $|A_n| \leq M\,q^n$ for some $M > 0$ and $q < 1$.

For the function L we found that

$$L(z) \approx \frac{z}{1-\frac{51}{350}\,z^2} + \frac{z^3/5.329...}{1-\frac{1009}{1400}\,z^2} + \frac{z^5/22.96...}{1-\ z^2} - \frac{z^7/806.9...}{1+\frac{59}{100}\,z^2} - \frac{z^9/259.3...}{1-\ z^2} + \frac{z^{11}/1039.7...}{1-\frac{1567}{2100}\,z^2} - \frac{z^{13}/4274.2...}{1+\ z^2} + \frac{z^{15}/16697.9...}{1+\ z^2} - \frac{z^{17}/73749.6...}{1-\alpha_9\,z^2} ..., \tag{8}$$

where the n-th coefficient is less than 4^{1-n} for $n \leq 9$. The method is the following branch and bound algorithm. Fix an integer m and consider k nested cycles for $\alpha_n, n = 1, ..., k$ ranging from -1 to 1 with step $2/m$. The bound is $A_{n+1} \leq 4^{-n}$ and if this is not fulfilled, the corresponding cycle continues with the next iteration, avoiding going into deeper levels. The algorithm works successfully up to $k = 7$. For (8), a modification was used to justify the coefficients to $k = 8$.

3. Consider the following lacunary representation

$$L(z) \approx \sum_{k=0}^{\infty} B_k \Big(\frac{z}{1-\beta_k z^2}\Big)^{4k+1}. \tag{9}$$

The parameters in (9) are uniquely determined, with the first several of them given by:

$$B_0 = 1, \quad \beta_0 = \frac{1}{3}; \qquad B_1 = \frac{4}{45} = \frac{1}{11.25}, \quad \beta_1 = \frac{5}{21} = 0.23809...;$$

$$B_2 = \frac{92}{63^2} = \frac{1}{43.14...}, \qquad \beta_2 = \frac{163}{759} = 0.21475...;$$

$$B_3 = \frac{22458728}{3015483471} = \frac{1}{134.2...}, \qquad \beta_3 = \frac{4150546877}{20339185545} = 0.20406...;$$

$$B_4 = \frac{1}{378.6...}, \qquad \beta_4 = 0.197876...;$$

$$B_5 = \frac{1}{1007.8...}, \qquad \beta_5 = 0.193803...;$$

Thus, we can formulate the following

Hypothesis 2. The representation (9) of $L(z)$ converges when z belongs to a certain disk centered at the origin.

Note that the analogous representation for $l(x)$ leads exactly to (5).

4. Let us consider the following combination of (2) and the above form:

$$L(z) \approx \frac{B_0}{1-b_0z^2}\left(\frac{z}{1-\beta_0z^2}\right) + \frac{B_1}{1-b_1z^2}\left(\frac{z}{1-\beta_1z^2}\right)^5 + \frac{B_2}{1-b_2z^2}\left(\frac{z}{1-\beta_2z^2}\right)^9 + \cdots . \tag{10}$$

In this form, keeping the lacunary property, we have a series of extra parameters in order to optimize the coefficients. Say if we choose $\{\beta_n\}$, the parameters $\{B_n\}$ and $\{b_n\}$ are uniquely determined, provided it does not appear division by zero. The following choice satisfy: $|\beta_n|, |b_n| \le 1$ for $n = 0, ..., 3$; $|B_n| \le 20^{-n}$ for $n = 0, ..., 4$; and provide a possibility for arbitrary small $|B_5|$ with $|\beta_4| \le 1$:
$\{\beta_n\}_0^3 = \{0.4254, 0.1427, 0.0238, 0.411\}$;
$\{b_n\}_0^3 = \{-0.092066..., 0.889557..., 0.925184..., -0.478074...\}$;
$\{B_n\}_0^4 = \{1, 1/20.1111..., -1/521.310..., 1/19118.7..., 1/161497.8...\}$.

The method is by considering the graphs of two consecutive B_n and B_{n+1} with respect to β_{n-1} and β_n in order to choose β_{n-1}. The graph of $B_n(\beta_{n-1})$ is a parabola and we introduce the notion *balanced choice* of the previous parameters if the graph intersects the abscissa for $\beta_{n-1} \in [-1, 1]$, that is if we can make $|B_n|$ arbitrarily small. But if we take $B_n = 0$ then B_{n+1} becomes undefined because of division by zero. This is clearly seen from the second graph of $B_{n+1}(\beta_{n-1}, \beta_n)$ which has infinite branches, at the places where $B_n = 0$. So, it is good to choose β_{n-1} close to these vertical asymptotes (the zeros of $B_n(\beta_{n-1})$) so that the corresponding section of the 3D graph (which is the planar graph for the next step) crosses the zero level. Actually, considering the 3D graphs is an auxiliary process, and we can avoid this. We can try several specific values of β_{n-1} close to the zeros of $B_n(\beta_{n-1})$ so that $|B_n|$ is small and the next graph of $B_{n+1}(\beta_n)$, $\beta_n \in [-1, 1]$ has zeros, i.e. the choice of β_{n-1} to be balanced. If, say, $B_n(\beta_{n-1})$ has two zeros in $[-1, 1]$, then it can happen to exist four appropriate areas for choosing β_{n-1}, on the both sides of the two zeros. An additional reasoning which helps the choice is the goal to keep the parameters $\{b_n\}$ in $[-1, 1]$. Then, the choice of β_{n-1} has to be such that $|b_{n-1}| \le 1$ and since the function $b_n(\beta_n) = A(\beta_{n-1})\beta_n + B(\beta_{n-1})$ is linear, it is easy to estimate in advance the range of b_n when $\beta_n \in [-1, 1]$.

A natural question is if there exists a balanced choice of $\{\beta_{n-1}\}$ for every $n \in \mathbb{N}$.

Revisiting example (9) considered as a particular case of (10) we make the following observations. The choice of $\{\beta_i\}_{i=0}^{n-1}$ is balanced up to $n = 30$, as the graphs of $B_{n+1}(\beta_n)$ (with specified previous $\{\beta_i\}$) have two roots in $[-1, 1]$ and the specific value for β_n in (9) is between the middle of them and the second root. It seems that the series $\{\beta_n\}$ has a limit around 0.17 and the ratios B_{n+1}/B_n belong to $(0, 1/2)$.

Other interesting choice is to specify β_n at the extreme point of the parabola $y = B_{n+1}(\beta_n)$. On the basis of calculations made up to $n = 30$, the situation appears to be very similar to the one described above, but now β_n is exactly the middle of the two roots of $B_{n+1}(\beta_n)$. Surprisingly, we observe that $b_n = \beta_n$ and seemingly this series tends to the same limit as above.

Clearly, there is much subjectivity in the approach described above, but it is not easy to avoid it. For example, if we use the least squares criterion $M_n = \lambda_n B_n^2 + (1-\lambda_n)B_{n+1}^2 \to \min$ then the subjectivity transfers to the choice of the λ-s. The function $M_n(\beta_{n-1}, \beta_n)$ usually has several local extrema and a decent optimization of the sequence $\{B_n\}_1^k$ needs considering of a tree of possibilities. Note that the attempt to manage the parameters by minimizing of the three term sums $\lambda_n B_n^2 + \mu_n B_{n+1}^2 + \nu_n B_{n+2}^2$ was not successful because of the complicatedness of this three variable function.

4. CONVERGENCE CONSIDERATIONS

Let us consider the representation (10) with equal parameters $\beta_n = p, n = 0, 1, 2, \ldots$. This form is motivated as a simple generalization of the lacunary variant of (2) (for $zL(z), x = z^2$), which hopefully will converge for certain p. We start the study of the series $\{B_n(p)\}$ and $\{b_n(p)\}$ with some particular examples. For $p = 0$ we have the second lacunary example from 3.1, while for $p = \frac{1}{4}$ we have

$$\begin{aligned}
\{B_n\} = \{&1, 1.0972...10^{-1}, 2.1442...10^{-2}, 4.5910...10^{-3}, 8.9862...10^{-4}, -7.5297...10^{-5},\\
&3.0626...10^{-3}, -3.8502...10^{-2}, 3.4662..., -1.2595...10^{3}, 1.6502...10^{6}, \ldots\};\\
\{b_n\} = \{&1/12, -0.15898..., -0.25539..., -0.20173..., 0.17256..., -6.7384..., -8.4293...,\\
&-17.216..., -33.759..., -64.545..., -122.64..., \ldots\}.
\end{aligned}$$

The behavior of this sequence is typical: For common values of p, in the beginning $|B_n|$ decreases like a geometric series, later on the decreasing slows down and changes to increasing and finally we observe again a rapid divergence to ∞. Slightly before the turning of $\{B_n\}$ it is preceded by breaking the restriction $|b_n| \le 1$. Especially, for $p = 0.17$ the decreasing lasts up to $n = 336$, when $B_{336} = 9.1654...10^{-119}$, and after that again $|B_n|$ goes to ∞. A natural question is whether there exist real values of p for which $\{B_n(p)\}_0^\infty$ is bounded. However, a numerical search for such values encounters some difficulties. For example, the above number was obtained by using long arithmetics and a precision of 200 decimal digits was not sufficient.

Usually we get the limit behavior $b_n \approx A.q^n$ and $B_n \approx (-1)^n B.q^{n^2-\alpha n}$ with $q > 1$. While A, B and α in the above empirical formulas depend on p, it is interesting that $q \approx 1.894$ is an absolute constant. Indeed, assume that the above relations hold as asymptotic equivalences "$\sim$" and $|q| > 1$. By (10) with $\beta_n = p$ we obtain the following system for the coefficients B_n and b_n:

$$\begin{aligned}
B_0 \sum_{i=0}^{2n} p^i b_0^{2n-i} + B_1 \sum_{i=0}^{2n-2} \tbinom{4+i}{i} p^i b_1^{2n-2-i} + B_2 \sum_{i=0}^{2n-4} \tbinom{8+i}{i} p^i b_2^{2n-4-i} + \cdots + B_n &= \frac{1}{4n+1}, \\
B_0 \sum_{i=0}^{2n+1} p^i b_0^{2n+1-i} + B_1 \sum_{i=0}^{2n-1} \tbinom{4+i}{i} p^i b_1^{2n-1-i} + B_2 \sum_{i=0}^{2n-3} \tbinom{8+i}{i} p^i b_2^{2n-3-i} + \cdots & \\
+ B_n(b_n + (4n+1)p) &= \frac{1}{4n+3}.
\end{aligned} \tag{11}$$

We observe that, for a sufficiently large n the back terms in (11) are significant, while the first terms are relatively small (we assume that $|p| < 1$). Also, the first summands $b_{n-j}^{2j+\delta}, \delta \in \{0, 1\}$ in the rear sums (for $i = 0$) are equivalent to the whole sums. For example, next to the last term in the left hand side of the first equation is $B_{n-1}(b_{n-1}^2 + (4n-3)pb_{n-1} + (2n-1)(4n-3)p^2) \sim B_{n-1}b_{n-1}^2$ for $n \to \infty$. That's why the terms containing p are negligible for $n \to \infty$ according to the assumption. Thus, for a sufficiently large n we come to the limit system

$$\begin{aligned}
&B_n.1 + B_{n-1}.b_{n-1}^2 + B_{n-2}.b_{n-2}^4 + \cdots + B_0.b_0^{2n} = o\big(B_n\big), \\
&B_n.b_n + B_{n-1}.b_{n-1}^3 + B_{n-2}.b_{n-2}^5 + \cdots + B_0.b_0^{2n+1} = o\big(B_n\big).
\end{aligned}$$

Substituting here the asymptotic relations for b_n and B_n we see that A^{2n} and $q^{\alpha n}$ are combined. Then, with $u = A^2 q^\alpha$, letting $n \to \infty$ we come to the following system

$$\begin{aligned}
&1 - uq^{-1^2} + u^2.q^{-2^2} - u^3.q^{-3^2} + \cdots = 0, \\
&1 - uq^{-1.2} + u^2.q^{-2.3} - u^3.q^{-3.4} + \cdots = 0.
\end{aligned}$$

We did not investigate this system for all real solutions, but considering truncated systems, which are algebraic, we found a series of real solutions that stabilizes to $(q, u) = (1.8947..., 6.1450...)$.

In order to understand better the behavior of the series $\{B_n\}$ and $\{b_n\}$ we consider first the truncated recurrence system

$$\begin{aligned}
&B_n + B_{n-1}\Big[b_{n-1}^2 + \tbinom{4n-3}{1} pb_{n-1} + \tbinom{4n-2}{2} p^2\Big] = \frac{1}{4n+1} \\
&B_n\big(b_n + (4n+1)p\big) + B_{n-1}\Big[b_{n-1}^3 + \tbinom{4n-3}{1} pb_{n-1}^2 + \tbinom{4n-2}{2} p^2 b_{n-1} + \tbinom{4n-1}{3} p^3\Big] = \frac{1}{4n+3}.
\end{aligned} \tag{12}$$

and its specification (with $p = 0$)

$$\begin{aligned}
B_n + B_{n-1}b_{n-1}^2 &= \frac{1}{4n+1} \\
B_n b_n + B_{n-1}b_{n-1}^3 &= \frac{1}{4n+3}.
\end{aligned} \tag{13}$$

The system (13) appears to have a similar behavior as (11) in qualitative sense, but in quantitative sense it is weaker. Depending on B_0 and b_0, in the general case the series fluctuates in the beginning and from some place on stabilizes to the asymptotic formulas $b_n \to \rho$, $|\rho| > 1$ and $B_n \sim C(-\rho^2)^n$. It is possible that the series terminates if some B_k vanishes and consequently $b_k = c/0$. We shall prove a "divergence criterion" which imply the asymptotic formulas (if they hold) for a concrete initial pair (B_0, b_0).

Consider first the case when $B_k = \varepsilon$ for a sufficiently small $|\varepsilon|$. Then, $b_k = c/\varepsilon$ and let us assume that $|c|$ is not very small, say $\varepsilon = o(c)$ for $\varepsilon(B_0, b_0) \to 0$ and $B_0 = const$. For the next terms, we find from (13)

$$B_{k+1} = \frac{1}{4k+5} - \frac{c^2}{\varepsilon} = -\Big(\frac{c}{\varepsilon}\Big)^2 B_k\Big(1 + O(\varepsilon)\Big)$$
$$b_{k+1} = \Big(\frac{1}{4k+7} - \frac{c^3}{\varepsilon^2}\Big)/B_{k+1} = b_k\Big(1 + O(\varepsilon)\Big).$$

Similarly, $B_{k+j} = \Big(-\dfrac{c^2}{\varepsilon^2}\Big)^j B_k\Big(1 + O(\varepsilon)\Big)$ and $b_{k+j} = b_k\Big(1 + O(\varepsilon)\Big)$ for every fixed $j \in \mathbb{N}$. Thus, if B_k happens to be very close to 0, then the asymptotic formulas take place immediately after k with a large ρ.

Proposition 1. *Let $\{(B_n, b_n)\}$ satisfy (13) and for a fixed k the conditions $|b_{k-1}| \geq r > 1$ and $|B_{k-1}b_{k-1}^2| \geq \dfrac{Q}{4k+1}$ with $Q > 3$ hold true. Then*

$$|b_k| \geq r - \frac{1+r}{Q-1} \quad \textit{and} \quad |B_k b_k^2| \geq \frac{r^2(Q-3)^2}{(4k+5)(Q-1)}.$$

Proof. From (13) it follows $|B_k| \geq |B_{k-1}b_{k-1}^2| - \frac{1}{4k+1}$ and $B_k \frac{b_k}{b_{k-1}} + \Big(\frac{1}{4k+1} - B_k\Big) = \frac{1/b_{k-1}}{4k+3}$. Therefore, $|B_k| \geq \frac{Q-1}{4k+1}$ and $|B_k| \cdot \Big|\frac{b_k}{b_{k-1}} - 1\Big| \leq \frac{1}{4k+1} + \frac{1/r}{4k+3} \leq \frac{1+1/r}{4k+1}$. As a consequence we have $\Big|\frac{b_k}{b_{k-1}} - 1\Big| \leq \frac{1+1/r}{Q-1}$. Hence, $\frac{b_k}{b_{k-1}} \geq 1 - \frac{1+1/r}{Q-1}$, and since the latter number is positive ($Q > 3$), we obtain that

$$|b_k| \geq |b_{k-1}|\Big(1 - \frac{1+1/r}{Q-1}\Big) \geq r\Big(1 - \frac{1+1/r}{Q-1}\Big) = r - \frac{1+r}{Q-1}$$

and

$$|B_k b_k^2| \geq \frac{Q-1}{4k+1}\Big(\frac{rQ-2r-1}{Q-1}\Big)^2 > \frac{r^2(Q-3)^2}{(4k+5)(Q-1)}.$$

□

The following assertion (which is a divergence criterion) makes use of the fact that for a sufficiently large Q, the estimates from Proposition 1 essentially repeat recursively and imply that $\{B_n\}$ increases at least as a geometric sequence.

Corollary 1. *Let $\{(B_n, b_n)\}$ satisfy (13) and for a given k the estimates $|b_k| \geq q + \varepsilon$ and $|B_k b_k^2| \geq \frac{Q}{4k+5}$ hold true, where $q > 1$, $\varepsilon > 0$ and $Q > 3$. If in addition $\varepsilon(Q-3) \geq \frac{q^2}{q-1}$, then for every $j \in \mathbb{N}_0$ we have*

$$|b_{k+j}| \geq q, \quad |B_{k+j} b_{k+j}^2| \geq \frac{(Q-3)q^{2j}}{4(k+j)+5} \quad \textit{and} \quad |B_{k+j+1}| \geq \frac{(Q-3)q^{2j}-1}{4(k+j)+5}.$$

Before proving Corollary 1, we will prove a technical lemma.

Lemma 1. *For given $q > 1$, $\varepsilon > 0$ and $Q > 3$ let us define the sequences $\{\varepsilon_j\}_0^\infty$ and $\{Q_j\}_0^\infty$ by $\varepsilon_0 = \varepsilon$, $(q + \varepsilon_{j+1}) = (q + \varepsilon_j) - \frac{1+(q+\varepsilon_j)}{Q_j - 1}$ and $Q_0 = Q$, $Q_{j+1} = (q+\varepsilon_j)^2 \frac{(Q_j-3)^2}{(Q_j-1)}$. If in addition $\varepsilon(Q-3) \geq \frac{q^2}{q-1}$, then for every $j \in \mathbb{N}_0$ the inequalities $\varepsilon_j \geq \frac{q^2}{(q-1)(Q_j-3)} > 0$ and $(Q_j - 3) \geq (Q-3)q^{2j} > 0$ hold true.*

Proof. Clearly, it is enough to prove the assertion only for $j = 1$, as for larger j it follows inductively. For brevity, set $\bar{Q} := Q - 3$.

We start with the proof of inequality $(Q_1 - 3) \geq (Q-3)q^2$. It is equivalent to

$$(q+\varepsilon)^2 \bar{Q}^2 \geq (\bar{Q}q^2 + 3)(\bar{Q} + 2).$$

In view of the additional assumption for ε, the above will follow from

$$q^2\bar{Q}^2 + \frac{2q^3}{q-1}\bar{Q} + \frac{q^4}{(q-1)^2} \geq (\bar{Q}q^2 + 3)(\bar{Q} + 2),$$

which is $\dfrac{2q^3}{q-1}\bar{Q} + \dfrac{q^4}{(q-1)^2} \geq (2q^2+3)\bar{Q} + 6$ and easily follows by termwise comparison of the summands in the left- and right-hand sides, taking into account that $q > 1$.

The inequality $\varepsilon_1 \geq \dfrac{q^2}{(q-1)(Q_1-3)}$, by the definitions and the just proved $Q_1 > 3$, is equivalent to

$$\Big[\varepsilon\Big(\frac{Q-2}{Q-1}\Big) - \frac{1+q}{Q-1}\Big] \cdot \Big[(q+\varepsilon)^2 \frac{(Q-3)^2}{Q-1} - 3\Big] \geq \frac{q^2}{q-1}.$$

It is not difficult one to verify that the first factor in the left-hand side is positive, as it is positive for ε replaced with its lower bound $\frac{q^2}{(q-1)(Q-3)}$. Therefore, the above inequality will hold true if it is true with $\varepsilon = \frac{q^2}{(q-1)(Q-3)}$, which is

$$\Big[\frac{q^2}{q-1} \cdot \frac{Q-2}{(Q-1)(Q-3)} - \frac{1+q}{Q-1}\Big] \cdot \Big[\Big(q + \frac{q^2}{(q-1)(Q-3)}\Big)^2 \frac{(Q-3)^2}{Q-1} - 3\Big] \geq \frac{q^2}{q-1}.$$

The latter is equivalent to the inequality

$$\big[q^2 + \bar{Q}\big] \cdot \big[q^2(q-1)^2\bar{Q}^2 + 2q^3(q-1)\bar{Q} + q^4 - 3(q-1)^2(\bar{Q}+2)\big] \geq q^2(q-1)^2(\bar{Q}+2)^2\bar{Q}$$

which after simplification takes the form $A\,\bar{Q}^2 + B\,\bar{Q} + C \ge 0$ with coefficients $A = (q-1)\big(2q^3 + (q-1)(q^4 - 4q^2 - 3)\big)$, $B = q^4 + 2(q-1)q^5 - (q-1)^2(7q^2+6)$ and $C = q^2(q^4 - 6(q-1)^2)$.

It is easy to verify that polynomials A, B and C are positive for $q > 1$. The positivity C follows from $q^4 - 6(q-1)^2 = (q-1)^4 + 4(q-1)^3 + 4(q-1) + 1 > 0$. To check that $A > 0$ we write $2q^3 + (q-1)(q^4 - 4q^2 - 3) = 2q^3 - 7(q-1) + (q-1)(q^2-2)^2$ and $2q^3 - 7(q-1) = 2(q-1)^3 + 6(q-1)^2 - (q-1) + 1 > 0$ since $q-1$ is majorized either by $(q-1)^2$ or by 1. Finally, to verify that $B > 0$ we rewrite it as $\big(q^4 - 6(q-1)^2\big) + q^2(q-1)\big(2q^3 - 7(q-1)\big)$, where the positivity of the both summands was already shown. The lemma is proved. $\square$

Proof of Corollary 1. Define the sequences $\{\varepsilon_j\}_0^\infty$ and $\{Q_j\}_0^\infty$ as in Lemma 1. The definitions are coherent with Proposition 1 so that (by induction) $|b_{k+j}| \ge q + \varepsilon_j$ and $|B_{k+j} b_{k+j}^2| \ge \frac{Q_j}{4(k+j)+5}$ for $j \ge 0$. The conditions of Proposition 1, $q + \varepsilon_j > 1$ and $Q_j > 3$, are ensured by Lemma 1 on the basis of the additional condition for ε and Q. Furthermore, the estimates from Lemma 1, $\varepsilon_j \ge 0$ and $(Q_j - 3) \ge (Q-3)q^{2j}$, imply the first two claimed estimates in the corollary. The third inequality is an elementary consequence from the second one and the first row of (13). $\square$

The next assertion claims that, essentially, the above lower estimates describe the asymptotical behavior of the series generated by (13).

Corollary 2. *Under the conditions of Corollary 1 the asymptotic relations $b_n \to \rho$, $|\rho| > 1$ and $B_n \sim C\,(-\rho^2)^n$ hold for $n \to \infty$, where $\rho = \rho(B_0, b_0)$ and $C = C(B_0, b_0)$.*

Proof. Denote $B_n b_n^2$ by M_n. Increasing if necessary the index k in Corollary 1, we may assume that $Q > 6$, hence $(4(k+j)+5)M_{k+j} > 3$ for every $j > 0$. Using (13) we obtain $B_{k+j+1} = -M_{k+j}\Big(1 - \frac{1}{(4(k+j)+5)M_{k+j}}\Big)$ and

$$b_{k+j+1} = \frac{-M_{k+j} b_{k+j} + \frac{1}{4(k+j)+7}}{B_{k+j+1}} = \frac{b_{k+j} - \frac{1}{(4(k+j)+7)M_{k+j}}}{1 - \frac{1}{(4(k+j)+5)M_{k+j}}} =: b_{k+j}\big(1 + \tau_{k+j}\big).$$

Then we have

$$|\tau_{k+j}| = \left| \frac{\frac{1}{(4(k+j)+5)M_{k+j}} - \frac{1}{(4(k+j)+7)b_{k+j}M_{k+j}}}{1 - \frac{1}{(4(k+j)+5)M_{k+j}}} \right| \le \frac{\Big(\frac{1}{4(k+j)+5} + \frac{q}{4(k+j)+7}\Big)\frac{1}{|M_{k+j}|}}{1 - \frac{1}{(4(k+j)+5)|M_{k+j}|}}$$
$$\le \frac{2}{(4(k+j)+5)|M_{k+j}|}\,\frac{1}{1 - 1/3} \le \frac{3}{(Q-3)q^{2j}} \le \frac{1/2}{q^{2j}}.$$

Therefore, for every natural $n \ge k$ and j the inequalities

$$\frac{b_{n+j}}{b_n} = \prod_{i=0}^{j-1}(1 + \tau_{n+i}) \ge \prod_{i=0}^{j-1}(1 - |\tau_{n+i}|) \ge \prod_{i=0}^{\infty}(1 - |\tau_{n+i}|) \ge 1 - \sum_{i=0}^{\infty}|\tau_{n+i}|$$

and

$$\frac{b_{n+j}}{b_n} = \prod_{i=0}^{j-1}(1+\tau_{n+i}) \le \prod_{i=0}^{j-1}(1+|\tau_{n+i}|) \le \prod_{i=0}^{\infty}(1+|\tau_{n+i}|) \le \exp\Big(\sum_{i=0}^{\infty}|\tau_{n+i}|\Big)$$

hold true. As a consequence, in view of the estimates for τ_m, we obtain

$$-\frac{(1/2)}{q^{2(n-k)}(1-q^{-2})} \le \frac{b_{n+j}}{b_n} - 1 \le \exp\Big(\frac{(1/2)}{q^{2(n-k)}(1-q^{-2})}\Big) - 1.$$

Since these bounds can be arbitrarily close to 0 for a sufficiently large n and $\{|b_m|\}$ is bounded (by the same inequalities with $n=k$), we conclude that the sequence $\{b_m\}$ is fundamental, and hence convergent to a limit ρ with $|\rho| \ge q > 1$.

In addition, letting j to infinity, we find the estimates

$$-\frac{1/2}{q^{2(n-k)}(1-q^{-2})} \le \frac{\rho}{b_n} - 1 \le \exp\Big(\frac{1/2}{q^{2(n-k)}(1-q^{-2})}\Big) - 1, \qquad n \ge k.$$

Or, we can simplify these to

$$\Big|\frac{\rho}{b_n} - 1\Big| \le \frac{c(q)}{q^{2(n-k)}}, \quad n \ge k,$$

where $c(q) = \max_{x\in[0,1]} \frac{1}{x}\Big[\exp\Big(\frac{x/2}{1-q^{-2}}\Big) - 1\Big]$. Then, with $b_n =: \rho/(1+\theta_n)$, the bound $|\theta_n| \le c(q)q^{2(k-n)}$ holds for $n \ge k$. Next, by the first equation in (13) it follows that

$$\frac{B_{n+1}}{B_n} = -b_n^2\Big(1 - \frac{1}{b_n^2(4n+5)B_n}\Big) = \frac{-\rho^2}{(1+\theta_n)^2}\Big(1 - \frac{1}{(4n+5)M_n}\Big).$$

Thus, for $n \ge k$ we have

$$B_n = B_k(-\rho^2)^{n-k}\prod_{j=0}^{n-k-1}\big(1+\theta_{k+j}\big)^{-2}\Big(1 - \frac{1}{(4(k+j)+5)M_{k+j}}\Big).$$

Finally, the estimates $|\theta_{k+j}| \le c(q)q^{-2j}$ and $\frac{1}{(4(k+j)+5)|M_{k+j}|} \le \frac{1}{(Q-3)q^{2j}}$ ensure the convergence of the infinite product $P := \prod_{j=0}^{\infty}\big(1+\theta_{k+j}\big)^{-2}\Big(1 - \frac{1}{(4(k+j)+5)M_{k+j}}\Big)$. (In view of Corollary 1 we have $B_n \ne 0$ and the all factors in P do not vanish.) Therefore, the partial product is asymptotically equivalent to its limit and we obtain $B_n \sim B_kP(-\rho^2)^{n-k} = C(-\rho^2)^n$. The proof is complete. □

Let us consider an example for application of Corollary 1. Let $B_0 = 1$ and $b_0 = 0$. By (13) we obtain: $B_1 = 1/5$, $b_1 = 5/7$; $B_2 = 4/441$, $b_2 = 153/77$; $B_3 = 0.04111...$, $b_3 = -0.10924...$; $B_4 = 0.05833...$, $b_4 = 0.90318...$; $B_5 = 3.4754...\times 10^{-5}$, $b_5 = 14.4152...$; $B_6 = 0.03277...$, $b_6 = -2.04617...$; $B_7 = -0.10275...$, $b_7 =$

$-3.04678...$; $B_8 = 0.98415...$, $b_8 = -2.92393...$; $B_9 = -8.38692...$, $b_9 = -2.93641...$; etc. It can be verified that the assumptions of Corollary 1 hold for $k = 7$ with $q = 2.8$, $\varepsilon = 0.2$ and $Q = 31$. Then, this particular sequence tends to infinity like a geometrical series and the lower bounds $|b_n| \geq 2.8$ and $|B_{n+1}| \geq \frac{28(2.8)^{2n-14}-1}{4n+5}$ hold for all $n \geq 7$.

Remark 1. It seems that there are bounded solutions of (13) even with $B_0 = 1$. We have not a strict proof but there is a particular candidate - the sequence with $B_0 = 1$ and $b_0 = b^*$, where $b^* \in (0.9512609, 0.9512610)$.

Let us turn our attention to the system (12). For $p \neq 0$ the usual limit behavior of the sequences defined by (12) is $B_n \sim C(p, b_0, B_0)(-4p^2)^n n![(n-1)!]^3$ and $b_n \sim -2pn^2$. A divergence criterion is given by the following

Proposition 2. *Let $\{(B_n, b_n)\}$ satisfy (12) and for certain $k \geq \frac{1}{2|p|}$ there holds $|B_k| \geq 1$. Then the sequence $\{|B_n|\}$ tends to infinity faster than any geometrical series.*

Proof. Let us set $L_n := b_n^2 + (4n+1)pb_n + \binom{4n+2}{2}p^2$. It is easily verified that $L_n \geq \frac{p^2}{4}(4n+1)(4n+3)$. Then by (12) and $|B_k| \geq 1$ we have

$$\begin{aligned} |B_{k+1}| = \; & \Big|\frac{1}{4k+5} - B_k L_k\Big| \geq \Big(L_k - \frac{1}{(4k+5)|B_k|}\Big)|B_k| \\ \geq \; & \Big(L_k - \frac{1}{(4k+5)}\Big)|B_k| \geq \Big(\frac{p^2}{4}(4k+1)(4k+3) - \frac{1}{(4k+5)}\Big)|B_k|. \end{aligned} \tag{14}$$

Now, the condition $k \geq \frac{1}{2|p|}$ imply that $(4k+1)^2 > \frac{4}{p^2}$ and by (14), $|B_{k+1}| > |B_k|$. It follows inductively that $|B_{n+1}| > |B_n| \geq 1$ for every $n \geq k$. Now, take an arbitrary $q > 1$. In view of the last inequality for B_n, we may assume that k is sufficiently large so that $\rho_k := \frac{p^2}{4}(4k+1)(4k+3) - \frac{1}{(4k+5)} \geq q$. Then (14) yields $|B_{k+1}| \geq q|B_k|$. Since ρ_k is increasing, we can prove by induction using (14) that $|B_{n+1}| \geq q|B_n|$ for every $n \geq k$. $\square$

Now we will prove that in the general case (except eventually for some special values of p) there is a choice of (B_0, b_0), such that the sequence $\{B_n\}$ is bounded. The basic observation is that the asymptotic formulas $B_n \sim \frac{6|\bar{y}|}{(4n)^3 p^2}$ and $b_n \sim \bar{y}(4np)$ are compatible with the system (12) if $\bar{y} \approx -0.62654$ is the unique real solution of the equation $y^3 + y^2 + \frac{1}{2}\,y + \frac{1}{6} = 0$. The next assertion states the existence of such type solutions of (12).

Proposition 3. *For every nonzero real number p there exist $k \in \mathbb{N}_0$ and $B_k, b_k \in \mathbb{R}$ such that the sequences $\{B_n\}$ and $\{b_n\}$ determined by (12) for $n \geq k$ satisfy $B_n = O(n^{-3})$ and $b_n = O(n)$ as $n \to \infty$.*

For the proof of this proposition we need the following auxiliary result.

Lemma 2. *Let (α, β) belongs to the domain $\mathcal{D} = \{(\alpha, \beta) \in \mathbb{R}^2 : \ |\alpha|, |\beta| \le \frac{1}{10}\}$. Then the equation $y^3 + y^2 + \frac{1+\alpha}{2}\, y + \frac{1+\beta}{6} = 0$ has a unique real solution $y(\alpha, \beta) \in (-0.75, -0.5]$ which is a Lipschitz function in $\mathcal{D}$. Moreover, if $(\alpha_i, \beta_i) \in \mathcal{D}$, $i = 1, 2$, then $|y(\alpha_2, \beta_2) - y(\alpha_1, \beta_1)| \le \frac{15}{8}|\alpha_2 - \alpha_1| + \frac{5}{6}|\beta_2 - \beta_1|$.*

Proof. Let $f(y) = y^3 + y^2 + \frac{1+\alpha}{2}\, y + \frac{1+\beta}{6}$. Then $f'(y) = 3y^2 + 2y + \frac{1+\alpha}{2}$ has a negative discriminant when $|\alpha| \le \frac{1}{10}$ and hence $f'(y) > 0$ for every $y \in \mathbb{R}$. Consequently, for $(\alpha, \beta) \in \mathcal{D}$ the equation $f(y) = 0$ has one real solution, which is denoted by $y(\alpha, \beta)$. Next, since $f(-\frac{3}{4}) \le \big(-\frac{3}{4}\big)^3 + \big(-\frac{3}{4}\big)^2 + \frac{0.9}{2}\big(-\frac{3}{4}\big) + \frac{1.1}{6} < 0$ and $f(-\frac{1}{2}) \ge \big(-\frac{1}{2}\big)^3 + \big(-\frac{1}{2}\big)^2 + \frac{1.1}{2}\big(-\frac{1}{2}\big) + \frac{0.9}{6} = 0$, then $y(\alpha, \beta) \in (-0.75, -0.5]$ provided $(\alpha, \beta) \in \mathcal{D}$.

For $(\alpha_0, \beta_0) = (0, 0)$ and fixed $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \mathcal{D}$ let us set $y_i := y(\alpha_i, \beta_i)$ and $f_i(y) := y^3 + y^2 + \frac{1+\alpha_i}{2}\, y + \frac{1+\beta_i}{6}$, $i = 0, 1, 2$. Then

$$\begin{aligned} 0 &= f_2(y_2) - f_1(y_1) = f_0(y_2) - f_0(y_1) + \frac{\alpha_2 y_2}{2} + \frac{\beta_2}{6} - \frac{\alpha_1 y_1}{2} - \frac{\beta_1}{6} \\ &= f_0'(\eta)(y_2 - y_1) + \frac{\alpha_2}{2}(y_2 - y_1) + \frac{\alpha_2 - \alpha_1}{2} y_1 + \frac{\beta_2 - \beta_1}{6} \end{aligned}$$

with some $\eta \in [y_1, y_2]$ (or $[y_2, y_1]$). Therefore,

$$|y_2 - y_1| \le \Big(\frac{|\alpha_2 - \alpha_1|}{2}|y_1| + \frac{|\beta_2 - \beta_1|}{6}\Big) \Big/ \Big|f_0'(\eta) + \frac{\alpha_2}{2}\Big|.$$

Using that $\eta \in [-\frac{3}{4}, -\frac{1}{2}]$ we obtain

$$f_0'(\eta) + \frac{\alpha_2}{2} = 3\eta^2 + 2\eta + \frac{1 + \alpha_2}{2} \ge 3\eta^2 + 2\eta + 0.45 \ge 0.2,$$

hence

$$|y_2 - y_1| \le \Big(\frac{0.75}{2}|\alpha_2 - \alpha_1| + \frac{1}{6}|\beta_2 - \beta_1|\Big) \times 5 = \frac{15}{8}|\alpha_2 - \alpha_1| + \frac{5}{6}|\beta_2 - \beta_1|.$$

The lemma is proved. □

Proof of Proposition 3. Let us define the sequences $\{(b_{n,i}, B_{n,i})\}_{n=k}^{\infty}$, for $i = 0, 1, 2, \ldots$ and $k \in \mathbb{N}$, which will be specified later, by the recurrence formulas:

$$b_{n,0} := (4n+1)p\bar{y}, \qquad B_{n,0} := \frac{6|\bar{y}|}{(4n+1)^2(4n+5)p^2};$$

$$\begin{aligned} b_{n,i} :\ & B_{n,i-1}\Big[b_{n,i}^3 + \tbinom{4n+1}{1} p b_{n,i}^2 + \tbinom{4n+2}{2} p^2 b_{n,i} + \tbinom{4n+3}{3} p^3\Big] \\ &\qquad = \frac{1}{4n+7} - B_{n+1,i-1}\big(b_{n+1,i-1} + (4n+5)p\big), \\ B_{n,i} :\ & B_{n,i}\Big[b_{n,i}^2 + \tbinom{4n+1}{1} p b_{n,i} + \tbinom{4n+2}{2} p^2\Big] = \frac{1}{4n+5} - B_{n+1,i-1}. \end{aligned}$$

Set $b_{n,i} := (1+\delta_{n,i})b_{n,i-1}$ and $B_{n,i} := (1+\Delta_{n,i})B_{n,i-1}$. We shall show that the relative distances $\delta_{n,i}$ and $\Delta_{n,i}$ decay (with i) like a geometrical series, from where it will follow that $(b_{n,i}, B_{n,i})$ converge to a certain limit as $i \to \infty$.

We estimate separately $\delta_{n,1}$ and $\Delta_{n,1}$. The ratio $X = X_n := b_{n,1}\big/((4n+1)p)$ is a solution of the equation $X^3+X^2+\frac{1}{2}\Big(1+\frac{1}{4n+1}\Big)X+\frac{1}{6}\Big(1+\frac{1}{4n+1}\Big)\Big(1+\frac{2}{4n+1}\Big) = A_0$, where $A_0 := \Big[\frac{1}{4n+7} - B_{n+1,0}\big(b_{n+1,0}+(4n+5)p\big)\Big]\Big/\Big[(4n+1)^3p^3B_{n,0}\Big]$. Let us set $\alpha = \frac{1}{4n+1}$ and $\beta = \beta_{n,1} = \frac{3}{4n+1} + \frac{2}{(4n+1)^2} - 6A_0$. In view of the definitions of $b_{n,0}$ and $B_{n,0}$ we have

$$\begin{aligned}|\beta| &\le \frac{3}{4n+1} + \frac{2}{(4n+1)^2} + \frac{4n+5}{4n+7}\cdot\frac{1}{(4n+1)|p\bar{y}|} + \frac{6(1+\bar{y})}{(4n+1)(4n+9)p^2}\\ &\le \frac{3+1.6/|p|}{4n+1} + \frac{2+2.25/p^2}{(4n+1)^2} =: \bar{\beta}_n.\end{aligned}$$

Now, choose k such that $\bar{\beta}_n \le \frac{1}{20}$ for $n \ge k$. Thus, $|\beta| \le \frac{1}{20}$ and $\frac{3}{4n+1} \le \frac{1}{20}$, i.e. $|\alpha| \le \frac{1}{60}$. Then, an application of Lemma 2 gives

$$|X-\bar{y}| = |y(\alpha,\beta) - y(0,0)| \le \frac{15}{8}\alpha + \frac{5}{6}|\beta| \le \frac{15/8}{60} + \frac{5/6}{20} \le 0.073\ .$$

Therefore, $X \in (-0.7, -0.553)$ and $|\delta_{n,1}| = \big|b_{n,1}/b_{n,0} - 1\big| = \big|X/\bar{y} - 1\big| < 0.118$.

Before estimating $\Delta_{n,1}$ we estimate

$$L_{n,1} = b_{n,1}^2+(4n+1)pb_{n,1}+\frac{1}{2}\Big(1+\frac{1}{4n+1}\Big)(4n+1)^2p^2 = (4n+1)^2p^2\Big(X^2+X+\frac{1+\alpha}{2}\Big).$$

Since $X \in (-0.7, -0.5)$ and $\alpha \in \big(0, \frac{1}{60}\big)$, then $L_{n,1}/(4n+1)^2p^2 \in (0.25, 0.3)$. This, in view of

$$B_{n,1} = \Big(\frac{1}{4n+5} - B_{n+1,0}\Big)\Big/L_{n,1} = B_{n,0}\Big(\frac{1}{6|\bar{y}|} - \frac{1/p^2}{(4n+5)(4n+9)}\Big)\Big/\Big[L_{n,1}/(4n+1)^2p^2\Big]$$

and $\frac{1/|p|}{4n+1} < 0.03$ (a consequence of $\bar{\beta}_n \le \frac{1}{20}$) implies that $B_{n,1}/B_{n,0} \in (0.883, 1.065)$ (the numerator belongs to $(0.265, 0.2661)$). Therefore, $|\Delta_{n,1}| \le 0.117$.

Our goal is to prove by induction that $|\delta_{n,i}|, |\Delta_{n,i}| \le 7^{-i}$. The above estimates prove this assertion for $i = 1$, and we assume that $i \ge 2$ by the end of the proof. Next, with $Y_i = Y_{n,i} := b_{n,i}\big/(4n+1)p$, we have

$$Y_i^3 + Y_i^2 + \frac{1}{2}\Big(1+\frac{1}{4n+1}\Big)Y_i + \frac{1}{6}\Big(1+\frac{1}{4n+1}\Big)\Big(1+\frac{2}{4n+1}\Big) = A_{i-1},$$

where $A_{i-1} := \Big[\frac{1}{4n+7} - B_{n+1,i-1}\big(b_{n+1,i-1}+(4n+5)p\big)\Big]\Big/\Big[(4n+1)^3p^3B_{n,i-1}\Big]$. As above, we set $\alpha = \frac{1}{4n+1}$ and $\beta = \beta_{n,i} = \frac{3}{4n+1} + \frac{2}{(4n+1)^2} - 6A_{i-1}$. Then

$$\begin{aligned}\beta_{n,i} - \beta_{n,i-1} =& \frac{-6/p^2}{(4n+1)^3}\Bigg[\frac{1/p}{4n+7}\Big(\frac{1}{B_{n,i-1}} - \frac{1}{B_{n,i-2}}\Big) \\ &- \Big(\frac{b_{n+1,i-1}}{p} + 4n+5\Big)\frac{B_{n+1,i-1}}{B_{n,i-1}} + \Big(\frac{b_{n+1,i-2}}{p} + 4n+5\Big)\frac{B_{n+1,i-2}}{B_{n,i-2}}\Bigg].\end{aligned}$$

Consequently,

$$|\beta_{n,i} - \beta_{n,i-1}| \le \frac{6/p^2}{(4n+1)^3}\left[\frac{\big(1/|p|\big)|K|}{(4n+7)B_{n,i-2}} + \frac{|b_{n+1,i-2}/p+4n+5|B_{n+1,i-2}|M|}{B_{n,i-2}}\right], \quad (15)$$

where $K = 1 - \frac{B_{n,i-2}}{B_{n,i-1}}$ and $M = 1 - \frac{b_{n+1,i-1}/p+4n+5}{b_{n+1,i-2}/p+4n+5}\cdot\frac{B_{n+1,i-1}}{B_{n+1,i-2}}\cdot\frac{B_{n,i-2}}{B_{n,i-1}}$. By induction, the following estimate for K holds true:

$$|K| = \Big|1 - \frac{1}{1+\Delta_{n,i-1}}\Big| = \frac{|\Delta_{n,i-1}|}{1+\Delta_{n,i-1}} \le \frac{49}{48}\cdot 7^{1-i}.$$

Here we have used $|\Delta_{n,i-1}| \le \frac{1}{49}$ for $i \ge 3$ and $|\Delta_{n,1}| \le 0.117$ for $i = 2$. Now, we estimate the factor M. By induction,

$$\frac{B_{n+1,i-1}}{B_{n+1,i-2}}\cdot\frac{B_{n,i-2}}{B_{n,i-1}} = \frac{1+\Delta_{n+1,i-1}}{1+\Delta_{n,i-1}} \in \Big[\frac{1-7^{1-i}}{1+7^{1-i}}, \frac{1+7^{1-i}}{1-7^{1-i}}\Big]$$

and let

$$\frac{b_{n+1,i-1}/p+4n+5}{b_{n+1,i-2}/p+4n+5} = 1 + \frac{b_{n+1,i-1} - b_{n+1,i-2}}{b_{n+1,i-2} + (4n+5)p} =: 1+\varepsilon\,.$$

Then $|\varepsilon| = |\delta_{n+1,i-1}|\Big/\Big|1 + \frac{(4n+5)p}{b_{n+1,i-2}}\Big|$ and we need to estimate the denominator. For $i = 2$, by definition it is $|1 + 1/\bar{y}| \approx 0.5961$ (hence $|\varepsilon| < 0.2$), while for $i \ge 3$ we have (see above)

$$Y^{-1}_{n+1,i-2} = \frac{(4n+5)p}{b_{n+1,i-2}} = \frac{b_{n+1,i-3}}{b_{n+1,i-2}}\,\frac{b_{n+1,i-4}}{b_{n+1,i-3}}\cdots\frac{b_{n+1,1}}{b_{n+1,2}}\,\frac{(4n+5)p}{b_{n+1,1}} = \Big[X\prod_{j=2}^{i-2}(1+\delta_{n+1,j})\Big]^{-1}.$$

By induction we conclude that

$$Y^{-1}_{n+1,i-2} \in \bigg(\Big[X\prod_{j=2}^{\infty}(1-7^{-j})\Big]^{-1}, \Big[X\prod_{j=2}^{\infty}(1+7^{-j})\Big]^{-1}\bigg) \subset \big(-1.853, -1.395\big),$$

where we have used $X = X_{n+1} \in (-0.7, -0.553)$ and $\prod_{j=2}^{\infty}(1+7^{-j}) \approx 1.02388$, $\prod_{j=2}^{\infty}(1-7^{-j}) \approx 0.97626$. Hence, $|\varepsilon| \le 2.54 \times 7^{1-i}$.

It follows from the above estimates that

$$M \in \Big[1 - \frac{1+7^{1-i}}{1-7^{1-i}}\Big(1 + 2.54\times 7^{1-i}\Big), 1 - \frac{1-7^{1-i}}{1+7^{1-i}}\Big(1 - 2.54\times 7^{1-i}\Big)\Big].$$

As a consequence, $|M| \le \frac{4.54+2.54\times 7^{1-i}}{1-7^{1-i}} 7^{1-i} \le 5.72\times 7^{1-i} \quad (i\ge 2)$.

The remaining factors in (15) we estimate by the induction. For $j=0,1$ we have

$$\frac{B_{n+j,i-2}}{B_{n+j,0}} = (1+\Delta_{n+j,1})\cdots(1+\Delta_{n+j,i-2}) \in \begin{cases} (0.883, 1.065), & \text{for } \ i=3 \\ (0.862, 1.091), & \text{for } \ i>3 \end{cases}.$$

From $Y^{-1}_{n+1,i-2} = \frac{(4n+5)p}{b_{n+1,i-2}} \in \big(-1.853, -1.395\big)$ we infer $(b_{n+1,i-2}/p+4n+5)/(4n+5) \in (0.283, 0.461)$, and the latter inclusion holds for $i=2$ as well. Then (15) implies

$$|\beta_{n,i} - \beta_{n,i-1}| \le$$

$$\frac{6/p^2}{(4n+1)^3\times 0.862 B_{n,0}}\Big[\frac{1/|p|}{(4n+7)}\times\frac{49}{48}\times 7^{1-i} + 0.461(4n+5)\times 1.091 B_{n+1,0}\times 5.72\times 7^{1-i}\Big]$$

$$\le \frac{7^{1-i}}{0.862(4n+1)}\Big[\frac{1}{|p\bar y|}\cdot\frac{49}{48}+\frac{2.877\times 6}{(4n+9)p^2}\Big] \le \Big[\frac{13.24/|p|}{4n+1}+\frac{140.2/p^2}{(4n+1)^2}\Big]\times 7^{-i}.$$

Hence, from $\frac{1/|p|}{4n+1} < 0.03$ we get $|\beta_{n,i}-\beta_{n,i-1}| \le 0.524\times 7^{-i} \quad (i\ge 2)$.

Inductively, similar inequalities hold for all $\{\beta_{n,j}\}_{j=2}^{i-1}$ and we conclude that

$$|\beta_{n,i}| \le |\beta_{n,1}| + \sum_{j=2}^{i}|\beta_{n,j}-\beta_{n,j-1}| < \frac{0.524\times 7^{-2}}{1-7^{-1}} + \frac{1}{20} < \frac{1}{10}.$$

Therefore, we can apply Lemma 2 to find for $i\ge 2$

$$|Y_i - Y_{i-1}| = |y(\alpha,\beta_{n,i}) - y(\alpha,\beta_{n,i-1})| \le \frac{15}{8}\times 0 + \frac{5}{6}|\beta_{n,i}-\beta_{n,i-1}| < 0.437\times 7^{-i}\,.$$

Also, Lemma 2 gives the estimate $Y_{i-1}\in(-0.75,-0.5]$ and we obtain

$$|\delta_{n,i}| = \big|Y_i/Y_{i-1}-1\big| = \big|Y_i - Y_{i-1}\big|/|Y_{i-1}| \le 0.874\times 7^{-i}, \quad i\ge 2.$$

In order to estimate $\Delta_{n,i}$ for $i\ge 2$ we use the identity

$$\Delta_{n,i} = \frac{B_{n,i}}{B_{n,i-1}} - 1 = \frac{1/(4n+5) - B_{n+1,i-1}}{L_{n,i}}\,\frac{L_{n,i-1}}{1/(4n+5) - B_{n+1,i-2}} - 1\,,$$

where $L_{n,i} := b^2_{n,i} + \binom{4n+1}{1} p b_{n,i} + \binom{4n+2}{2}p^2$. Clearly,

$$\begin{aligned} |L_{n,i} - L_{n,i-1}| &= |b_{n,i} - b_{n,i-1}|\times|b_{n,i}+b_{n,i-1}+(4n+1)p| \\ &= |b_{n,i-1}\,\delta_{n,i}|\times(4n+1)|p|\times|Y_i+Y_{i-1}+1|. \end{aligned}$$

Recalling that $Y_0 = \bar y$ and $Y_1 = X \in (-0.7,-0.553)$ we can refine the estimate $Y_i \in (-0.75,-0.5]$ by $|Y_i| \le |X|+|Y_1-Y_2|+\cdots+|Y_{i-1}-Y_i| < 0.7+\frac{0.437\times 7^{-2}}{1-7^{-1}} < 0.711$, i.e. $Y_i\in(-0.711,-0.5]$. The same holds for Y_{i-1} as well. Hence, $|Y_i+Y_{i-1}+1| < 0.422$ and we get

$$\Big|\frac{L_{n,i}}{L_{n,i-1}} - 1\Big| \le 0.369\times 7^{-i}\,\frac{|b_{n,i-1}(4n+1)p|}{|b^2_{n,i-1}+(4n+1)p b_{n,i-1}+\binom{4n+2}{2}p^2|} < \frac{0.369\times 7^{-i}|z|}{z^2+z+1/2},$$

where $z = \frac{b_{n,i-1}}{(4n+1)p} = Y_{i-1}$. The last expression we estimate by

$$\frac{|z|}{z^2 + z + 1/2} = \frac{1}{|z| - 1 + \frac{1}{2|z|}} =: \frac{1}{g(|z|)} \le \frac{1}{\min_{t>0}\, g(t)} = \sqrt{2} + 1\,.$$

Therefore, $\left|\frac{L_{n,i}}{L_{n,i-1}} - 1\right| < 0.891 \times 7^{-i}$. Next, let

$$N := \frac{1/(4n+5) - B_{n+1,i-1}}{1/(4n+5) - B_{n+1,i-2}} = 1 - \frac{B_{n+1,i-1} - B_{n+1,i-2}}{1/(4n+5) - B_{n+1,i-2}} = 1 - \frac{B_{n+1,i-2}\,\Delta_{n+1,i-1}}{1/(4n+5) - B_{n+1,i-2}}.$$

We found above $\frac{B_{n+1,i-2}}{B_{n+1,0}} \in (0.862, 1.091)$, hence $(4n+5)B_{n+1,i-2} \le \frac{1.091\times 6|\bar{y}|}{(4n+5)(4n+9)p^2}$ $< 4.102 \times 0.03^2 < 0.0037$. As a consequence, $\frac{B_{n+1,i-2}}{1/(4n+5) - B_{n+1,i-2}} < 0.004$. Therefore, $|1 - N| < 0.004 \times 7^{1-i} = 0.028 \times 7^{-i}$ and, finally,

$$|\Delta_{n,i}| = \left|\frac{N}{L_{n,i}/L_{n,i-1}} - 1\right| \le \frac{(0.028 + 0.891)7^{-i}}{1 - 0.891 \times 7^{-i}} \le 0.937 \times 7^{-i}, \quad i \ge 2.$$

The claimed estimates $|\delta_{n,i}|, |\Delta_{n,i}| \le 7^{-i}$ are proved.
These estimates imply that the sequences

$$b_{n,i} = b_{n,0} \prod_{j=1}^{i} (1 + \delta_{n,j})\,, \quad B_{n,i} = B_{n,0} \prod_{j=1}^{i} (1 + \Delta_{n,j})$$

converge as $i \to \infty$ to certain limits b_n^* and B_n^*, $n \ge k$. By the definitions of $\{(b_{n,i}, B_{n,i})\}_{n=k}^{\infty}$ it follows that the limit sequences satisfy (12). In addition, $b_n^*/b_{n,0}$ and $B_n^*/B_{n,0}$ are bounded, i.e. $b_n^* = O(np)$ and $B_n^* = O(1/n^3p^2)$. ☐

Thus we proved the existence of a solution $\{(B_n, b_n)\}$ of (12) with bounded B_n and $b_n = O(n)$ starting from a certain index $k(p)$. It is easily seen that (12) considered as a system for (B_{n-1}, b_{n-1}) is solvable in $\mathbb{R}^2$ provided $\frac{1}{4n+1} - B_n \ne 0$. Then, with the exception of some very special values for p, we can complete the obtained bounded sequence to the starting values $(B_0^*(p), b_0^*(p))$. For example, when $p = \frac{1}{2}$, the condition $\beta_k \le \frac{1}{20}$ is fulfilled for $k = 32$ and the values $(B_{32}^*, b_{32}^*) =$ $(6.7280929... \times 10^{-6}, -40.023137...)$ allow to complete uniquely the sequence up to $(B_0^*, b_0^*) = (0.28687201..., 0.34268557...)$ $(b_n^* < 0, n \ge 1)$. Note that in contrast to the backward calculations, which are stable, in order to get the above values for (B_{32}^*, b_{32}^*) starting from (B_0^*, b_0^*), the latter have to be given with at least 100 decimal digits.

Remark 2. In the special case $B_0 = 1$, $b_0 = \frac{1}{3} - p$, which is of interest for us (see (11)), it seems that there is no real p which determines a bounded sequence $\{B_n\}$ satisfying (12). This claim is based on exhaustive computer experiments. For example: when $|p| \ge \frac{5}{4}$, $|B_1| = |\frac{1}{5} - [(\frac{1}{3} - p)^2 + p(\frac{1}{3} - p) + p^2]| > 1$ and by

Proposition 2, $|B_n| \to \infty$; when $p \in [0.04277, 0.0428]$, the graph of $\log|B_{23}(p)|$ is clearly positive and Proposition 2 implies $|B_n| \to \infty$ in this case, too.

In view of the above results it is reasonable to consider lacunary transformations of $L(z)$ depending on two parameters. We formulate the following

Hypothesis 3. There is a choice of the real parameters p and q such that the representation

$$L(z) \approx qt + \sum_{n=0}^{\infty} \frac{B_n t^{4n+1}}{1 - b_n z^2}, \quad t = \frac{z}{1 - pz^2}$$

has coefficients satisfying $B_n = O(\rho^n)$ for some $\rho > 0$.

There is even some reason to expect the validity of Hypothesis 3 for $p = 0$, i.e. when $t = z$. This is because B_n is a rational function of q, whose numerator is an odd degree polynomial, and hence for every n there are values of q producing arbitrarily small B_n.

Of course, the magnitude of b_n is also important for the convergence of the series. If $\{B_n\}$ is bounded, but $\{b_n\}$ is not, then still it is enough b_n to be negative for $n \geq n_0$ and the convergence will hold in a real neighborhood of $z = 0$.

For the system (11) we will consider theoretically only the case $p = 0$. We can prove the following divergence criterion.

Proposition 4. *Assume that the sequences $\{B_n\}_0^\infty$ and $\{b_n\}_0^\infty$ satisfy the system*

$$\begin{array}{rcl} B_n + B_{n-1}b_{n-1}^2 + B_{n-2}b_{n-2}^4 + \cdots + B_0 b_0^{2n} & = & d_n^{(0)} \\ B_n b_n + B_{n-1}b_{n-1}^3 + B_{n-2}b_{n-2}^5 + \cdots + B_0 b_0^{2n+1} & = & d_n^{(1)}, \end{array} \tag{16}$$

where $|d_n^{(j)}| \leq 1$ for $j = 0, 1$. Let us denote $Y_n := b_n/b_{n-1}$, $Z_n := -B_n/(B_{n-1}b_{n-1}^2)$, $X_n := Y_n Z_n$ and $(\tilde{Z}^*, \tilde{X}^*) := (0.30834705, 0.58425448)$. Then the conditions

$$\begin{array}{l} |Z_{k-i} - \tilde{Z}^*|,\ |X_{k-i} - \tilde{X}^*| \leq r, \quad i = 0, 1, 2, 3\,; \\ |Z_{k-4} - \tilde{Z}^*|,\ |X_{k-4} - \tilde{X}^*| \leq 5 \times 10^{-5}\,; \\ \Big(1 + \sum\limits_{i=0}^{k-6} \big|B_i b_i^{2(k+1-i)+j}\big|\Big) \Big/ \big|B_k b_k^{2+j}\big| \leq 10^{-8}, \quad j = 0, 1\,; \\ |b_{k-1}| = \max\limits_{i \leq k-1} |b_i| \geq 1, \end{array} \tag{17}$$

for a given $k \geq 5$ and $r = 10^{-6}$ imply that $|B_n|$ tends to infinity faster than any geometrical series and $|b_n| \to \infty$.

Proof. We first change the variables and introduce vector notations. It is not difficult to verify that (16) is equivalent to

$$\begin{array}{rcl} 1 - 1/Z_n + 1/(Z_n Z_{n-1} Y_{n-1}^2) - 1/(Z_n Z_{n-1} Z_{n-2} Y_{n-1}^2 Y_{n-2}^4) + \cdots & = & d_n^{(0)}/B_n \\ 1 - 1/(Z_n Y_n) + 1/(Z_n Z_{n-1} Y_n Y_{n-1}^3) - 1/(Z_n Z_{n-1} Z_{n-2} Y_n Y_{n-1}^3 Y_{n-2}^5) + \cdots & = & d_n^{(1)}/B_n b_n, \end{array}$$

which is

$$
\begin{aligned}
Z_n = & \ 1 - \frac{Z_{n-1}}{X_{n-1}^2} + \frac{Z_{n-1}Z_{n-2}^3}{X_{n-1}^2X_{n-2}^4} - \frac{Z_{n-1}Z_{n-2}^3Z_{n-3}^5}{X_{n-1}^2X_{n-2}^4X_{n-3}^6} + \cdots - \frac{d_n^{(0)}}{B_{n-1}b_{n-1}^2} \\
X_n = & \ 1 - \frac{Z_{n-1}^2}{X_{n-1}^3} + \frac{Z_{n-1}^2Z_{n-2}^4}{X_{n-1}^3X_{n-2}^5} - \frac{Z_{n-1}^2Z_{n-2}^4Z_{n-3}^6}{X_{n-1}^3X_{n-2}^5X_{n-3}^7} + \cdots - \frac{d_n^{(1)}}{B_{n-1}b_{n-1}^3},
\end{aligned} \tag{18}
$$

where the sums are expanded to Z_1 and X_1, respectively. In this way we reduce the problem to the proof that the stationary point near $(\tilde{Z}^*, \tilde{X}^*)$ is stable. Indeed, then it will follow that, for $n \to \infty$, $b_n \approx A.(Y^*)^n$, where $Y^* \approx \tilde{X}^*/\tilde{Z}^* \approx 1.8948$ and $|B_{n+1}/B_n| \approx \tilde{Z}^* b_n^2 \to \infty$.

To prove the stability of the stationary point (Z^*, X^*) of (18) we use the approach based on fixed point theorems (see e.g. [8] and the references therein). Note that, formally, we will not use the existence of (Z^*, X^*).

In what follows we care mainly for the impact of the first four summands in (18) (excluding 1) while the remainder we estimate with less precision. So, let us denote $\overline{V}_n := (Z_n, X_n, Z_{n-1}, X_{n-1}, Z_{n-2}, X_{n-2}, Z_{n-3}, X_{n-3})^T$, then (18) becomes $\overline{V}_n = \bar{f}(\overline{V}_{n-1}) + \bar{\theta}_n$, or more precisely,

$$
\left|
\begin{array}{l}
\overline{V}_n(1) = \varphi(\overline{V}_{n-1}) + \epsilon_n \\
\overline{V}_n(2) = \psi(\overline{V}_{n-1}) + \delta_n \\
\overline{V}_n(i) = \overline{V}_{n-1}(i-2), \qquad i = 3, \ldots, 8,
\end{array}
\right.
$$

where

$$
\varphi(z_1, x_1, \ldots, z_4, x_4) := \tilde{a}^* + \sum_{i=1}^{4} (-1)^i z_1 \ldots z_i^{2i-1}/(x_1^2 \ldots x_i^{2i}),
$$

$$
\psi(z_1, x_1, \ldots, z_4, x_4) := \tilde{b}^* + \sum_{i=1}^{4} (-1)^i z_1^2 \ldots z_i^{2i}/(x_1^3 \ldots x_i^{2i+1}).
$$

Here,

$$
\tilde{a}^* = 1 + \sum_{i=5}^{\infty} (-1)^i (\tilde{Z}^*)^{i^2}/(\tilde{X}^*)^{i(i+1)} \quad \text{and} \quad \tilde{b}^* = 1 + \sum_{i=5}^{\infty} (-1)^i (\tilde{Z}^*)^{i(i+1)}/(\tilde{X}^*)^{i(i+2)}
$$

are approximations of the remainders of the sums in (18) when (Z_n, X_n) approaches the stationary point (Z^*, X^*). Assuming convergence of $\{(Z_n, X_n)\}$, the residuals ϵ_n and δ_n will become very small, but do not tend exactly to 0, because of the difference between $(\tilde{Z}^*, \tilde{X}^*)$ and (Z^*, X^*).

Our next step is to prove that conditions (17), but with $k = n$ and $r = 2 \times 10^{-5}$, imply the representation

$$
(\overline{V}_{n+1} - \tilde{V}^*) = \tilde{J}.(\overline{V}_n - \tilde{V}^*) + \bar{\varepsilon}_n, \tag{19}
$$

where $\tilde{V}^* = (\tilde{Z}^*, \tilde{X}^*, \ldots, \tilde{Z}^*, \tilde{X}^*) \in \mathbb{R}^8$, $\tilde{J}$ is a given approximation of the Jacobi matrix $\tilde{J}^* = \frac{D(\bar{f})}{D(\overline{V})}$ calculated at $\tilde{V}^*$ and $||\bar{\varepsilon}_n||_\infty \leq \varepsilon := 10^{-7}$.

Before doing this, we adopt the convention (for this proof only) that $||\cdot|| := ||\cdot||_\infty$ and $\alpha \approx \beta$ will mean that β is the rounded value of α to the corresponding decimal digit. For example $\alpha \approx -1.230 \times 10^{-5}$ means $|\alpha + 1.23 \times 10^{-5}| \leq \frac{1}{2} \times 10^{-8}$.

The Jacobian $D(\varphi, \psi, v_1, ..., v_6)/D(v_1, ..., v_8)$, $\bar{v} = (z_1, ..., x_4)$, at the point $\tilde{V}^*$, is calculated to be $\tilde{J}^* \approx \tilde{J} := (g_1, g_2, e_1, e_2, e_3, e_4, e_5, e_6)^T$, where

$$g_1 = (-2.2431, 2.3676, 2.0593, -1.4491, -0.2532, 0.1604, 0.0071, -0.0043),$$

$$g_2 = (-2.6966, 2.1347, 0.7911, -0.5219, -0.0451, 0.0278, 0.0006, -0.0004),$$

and e_j is the unit row vector in $\mathbb{R}^8$ whose j-th component equals 1.

It is important that the spectral radius $\rho(\tilde{J}) \approx 0.2539$ is less than 1. This means that the iterations of (19) will remain bounded provided the perturbation is sufficiently small. Now we start with the estimation of $\bar{\varepsilon}_n$.
We have $\overline{V}_{n+1} = \bar{f}(\overline{V}_n) + \bar{\theta}_{n+1}$ and set $\bar{\varepsilon}^1 := \bar{\theta}_{n+1} = (\epsilon_{n+1}, \delta_{n+1}, 0, ..., 0)^T$.

Next we justify the approximation $\bar{f}(\tilde{V}^*) \approx \tilde{V}^*$ by introducing $\bar{\varepsilon}^2 = \bar{f}(\tilde{V}^*) - \tilde{V}^*$, hence $(\overline{V}_{n+1} - \tilde{V}^*) = \bar{f}(\overline{V}_n) - \bar{f}(\tilde{V}^*) + \bar{\varepsilon}^1 + \bar{\varepsilon}^2$.

Applying Taylor's formula to the second order around $\tilde{V}^*$ we get

$$\bar{f}(\overline{V}_n) = \bar{f}(\tilde{V}^*) + \tilde{J}^*.(\overline{V}_n - \tilde{V}^*) + \frac{1}{2}\bar{Q}(\overline{V}_n - \tilde{V}^*),$$

where $\bar{Q}(\overline{V})$ is a vector whose components are quadratic forms of $\overline{V}$ and more precisely $\bar{Q}_1(\overline{V}) = \sum_{i,j} \frac{\partial^2 \varphi}{\partial v_i \partial v_j}(\bar{\eta}_1) V_i V_j$, $\bar{\eta}_1 \in [\tilde{V}^*, \overline{V}_n]$, $\bar{Q}_2(\overline{V}) = \sum_{i,j} \frac{\partial^2 \psi}{\partial v_i \partial v_j}(\bar{\eta}_2) V_i V_j$, $\bar{\eta}_2 \in [\tilde{V}^*, \overline{V}_n]$ and $\bar{Q}_i(\overline{V}) = 0$ for $i = 3, \ldots, 8$. We denote $\frac{1}{2}\bar{Q}(\overline{V}_n - \tilde{V}^*)$ by $\bar{\varepsilon}^3$ and then $(\overline{V}_{n+1} - \tilde{V}^*) = \tilde{J}^*.(\overline{V}_n - \tilde{V}^*) + \sum_{i=1}^3 \bar{\varepsilon}^i$.

Finally, the Jacobian $\tilde{J}^*$ is calculated approximately, hence with $\Delta\tilde{J} := \tilde{J}^* - \tilde{J}$ and $\bar{\varepsilon}^4 := \Delta\tilde{J}.(\overline{V}_n - \tilde{V}^*)$ we arrive at

$$(\overline{V}_{n+1} - \tilde{V}^*) = \tilde{J}.(\overline{V}_n - \tilde{V}^*) + \sum_{i=1}^4 \bar{\varepsilon}^i.$$

The estimation of $\bar{\varepsilon}^4$ is easy: we have

$$||\bar{\varepsilon}^4|| \leq ||\Delta\tilde{J}||.||\overline{V}_n - \tilde{V}^*|| \leq 8 \cdot \frac{1}{2} \times 10^{-4} r = 8 \times 10^{-9}.$$

In order to estimate $||\bar{\varepsilon}^3|| = \dfrac{1}{2} \max\limits_{i=1,2} |\bar{Q}_i(\overline{V}_n - \tilde{V}^*)| \leq \max\limits_{i=1,2} ||\bar{Q}_i|| \cdot r^2/2$ we use the obvious inequality: $||\sum\limits_{i,j} a_{i\,j} v_i v_j||_\infty := \max\limits_{||\bar{v}||=1} \Big|\sum\limits_{i,j} a_{i\,j} v_i v_j\Big| \leq \sum\limits_{i,j} |a_{i\,j}|$.

The quadratic form $\bar{Q}_1$ has coefficients $\frac{\partial^2\varphi}{\partial v_i \partial v_j}(\bar\eta_1) = \sum_{k=1}^4 (-1)^k \frac{\partial^2\varphi_k}{\partial v_i \partial v_j}(\bar\eta_1)$, where $\varphi_k(\bar v) := \varphi_k(z_1, x_1, ..., z_4, x_4) = z_1...z_k^{2k-1}/(x_1^2...x_k^{2k})$ (for $k \le 4$). Since φ_k (and ψ_k below) has the form $z_1^{\alpha_1}...z_k^{\alpha_k}x_1^{-\beta_1}...x_k^{-\beta_k}$, $\alpha_i, \beta_i \in \mathbb{N}$ we can use the general estimate

$$\begin{aligned}\sum_{i,j}\Big|\frac{\partial^2\varphi_k}{\partial v_i\partial v_j}(\bar\eta_1)\Big| &\le 2\, z_+^{\sum\alpha_i} x_-^{-\sum\beta_i}\Bigg[\frac{\sum_{i<j}\alpha_i\alpha_j + \sum_i \frac{\alpha_i(\alpha_i-1)}{2}}{z_+^2} \\ &\qquad + \frac{\sum_{i<j}\beta_i\beta_j + \sum_i \frac{\beta_i(\beta_i+1)}{2}}{x_-^2} + \frac{\sum_{i,j}\alpha_i\beta_j}{z_+x_-}\Bigg] \\ &= z_+^{\sum\alpha_i}x_-^{-\sum\beta_i}\Big[\Big(\frac{\sum\alpha_i}{z_+} + \frac{\sum\beta_i}{x_-}\Big)^2 - \frac{\sum\alpha_i}{z_+^2} + \frac{\sum\beta_i}{x_-^2}\Big],\end{aligned}$$

where $z_+(x_-)$ is an upper(a lower) bound of the odd(even) components of $\bar\eta_1$. Then, from $\bar\eta_1 \in [\tilde V^*, \overline{V}_n]$ and $||\overline{V}_n - \tilde V^*|| \le r$ it follows that $||\bar\eta_1 - \tilde V^*|| \le r$. Hence, we can take $z_+ = 0.3084 > \tilde Z^* + r$ and $x_- = 0.5842 < \tilde X^* - r$. So, for $k = 1,2,3,4$ we have
$\bar\alpha = (1)$, $\bar\beta = (2) \Rightarrow \sum_{i,j}\Big|\frac{\partial^2\varphi_1}{\partial v_i\partial v_j}(\bar\eta_1)\Big| \le \frac{z_+}{x_-^2}\Big[\frac{6}{x_-^2} + \frac{4}{z_+x_-}\Big] < 36$;
$\bar\alpha = (1,3)$, $\bar\beta = (2,4) \Rightarrow \sum_{i,j}\Big|\frac{\partial^2\varphi_2}{\partial v_i\partial v_j}(\bar\eta_1)\Big| \le \frac{z_+^4}{x_-^6}\Big[\Big(\frac{4}{z_+} + \frac{6}{x_-}\Big)^2 - \frac{4}{z_+^2} + \frac{6}{x_-^2}\Big] < 117.4$;
$\bar\alpha = (1,3,5)$, $\bar\beta = (2,4,6) \Rightarrow \sum_{i,j}\Big|\frac{\partial^2\varphi_3}{\partial v_i\partial v_j}(\bar\eta_1)\Big| \le \frac{z_+^9}{x_-^{12}}\Big[\Big(\frac{9}{z_+} + \frac{12}{x_-}\Big)^2 - \frac{9}{z_+^2} + \frac{12}{x_-^2}\Big] < 39$;
$\bar\alpha = (1,3,5,7)$, $\bar\beta = (2,4,6,8) \Rightarrow \sum_{i,j}\Big|\frac{\partial^2\varphi_4}{\partial v_i\partial v_j}(\bar\eta_1)\Big| \le \frac{z_+^{16}}{x_-^{20}}\Big[\Big(\frac{16}{z_+} + \frac{20}{x_-}\Big)^2 - \frac{16}{z_+^2} + \frac{20}{x_-^2}\Big] < 2.3$.

As a consequence, $||\bar Q_1|| \le \sum_{i,j}\Big|\frac{\partial^2\varphi}{\partial v_i\partial v_j}(\bar\eta_1)\Big| \le 195$.

Analogously, $\bar Q_2$ has coefficients $\frac{\partial^2\psi}{\partial v_i\partial v_j}(\bar\eta_2) = \sum_{k=1}^4(-1)^k\frac{\partial^2\psi_k}{\partial v_i\partial v_j}(\bar\eta_2)$, where $\psi_k(\bar v) := \psi_k(z_1, x_1, ..., z_4, x_4) = z_1^2...z_k^{2k}/(x_1^3...x_k^{2k+1})$ and hence $||\bar Q_2||$ is estimated by the sum of

$$\sum_{i,j}\Big|\frac{\partial^2\psi_1}{\partial v_i\partial v_j}(\bar\eta_2)\Big| < 59;\ \sum_{i,j}\Big|\frac{\partial^2\psi_2}{\partial v_i\partial v_j}(\bar\eta_2)\Big| < 68;\ \sum_{i,j}\Big|\frac{\partial^2\psi_3}{\partial v_i\partial v_j}(\bar\eta_2)\Big| < 10;\ \sum_{i,j}\Big|\frac{\partial^2\psi_4}{\partial v_i\partial v_j}(\bar\eta_2)\Big| < 1.$$

Thus, $||\bar Q_2|| < 138$ and therefore $||\bar\varepsilon^3|| \le \max\big(||\bar Q_1||, ||\bar Q_2||\big)r^2/2 < 4\times 10^{-8}$.

Next, we have

$$\begin{aligned}\bar\varepsilon^2 &= \big(\varphi(\tilde V^*) - \tilde Z^*, \psi(\tilde V^*) - \tilde X^*, 0, ..., 0\big)^T \\ &= \Big(\sum_{i=0}^{\infty}(-1)^i\frac{(\tilde Z^*)^{i^2}}{(\tilde X^*)^{i(i+1)}} - \tilde Z^*, \sum_{i=0}^{\infty}(-1)^i\frac{(\tilde Z^*)^{i(i+1)}}{(\tilde X^*)^{i(i+2)}} - \tilde X^*, 0, ..., 0\Big)^T \\ &\approx \big(-8.0\times 10^{-9}, -8.1\times 10^{-9}, 0, ..., 0\big)^T,\end{aligned}$$

and the Leibnitz type series are easy to estimate, yielding $||\bar\varepsilon^2|| < 10^{-8}$.

For an estimate of $\overline{\varepsilon}^1 = (\epsilon_{n+1}, \delta_{n+1}, 0, ..., 0)^T$ we write

$$\begin{aligned}
\epsilon_{n+1} &= -\frac{Z_n Z_{n-1}^3 Z_{n-2}^5 Z_{n-3}^7 Z_{n-4}^9}{X_n^2 X_{n-1}^4 X_{n-2}^6 X_{n-3}^8 X_{n-4}^{10}} + - \cdots \text{ (to } X_1) - \frac{d_{n+1}^{(0)}}{B_n b_n^2} - \sum_{i=5}^{\infty} (-1)^i \frac{(\tilde{Z}^*)^{i^2}}{(\tilde{X}^*)^{i(i+1)}} \\
&= \Big[-\varphi_5(Z_n, \ldots, X_{n-4}) + \frac{(\tilde{Z}^*)^{25}}{(\tilde{X}^*)^{30}} \Big] + \Big[\cdots \Big] - \sum_{i=6}^{\infty} (-1)^i \frac{(\tilde{Z}^*)^{i^2}}{(\tilde{X}^*)^{i(i+1)}} \\
&=: A + B + C.
\end{aligned}$$

$$|A| = \big|\varphi_5(Z_n, ..., X_{n-4}) - \frac{(\tilde{Z}^*)^{25}}{(\tilde{X}^*)^{30}}\big| \leq \max\Big\{ \big|\frac{(z_+)^{25}}{(x_-)^{30}} - \frac{(\tilde{Z}^*)^{25}}{(\tilde{X}^*)^{30}}\big|, \big|\frac{(z_-)^{25}}{(x_+)^{30}} - \frac{(\tilde{Z}^*)^{25}}{(\tilde{X}^*)^{30}}\big| \Big\},$$

where z_+ and x_- are as above while $z_- := 0.30829$ and $x_+ := 0.58431$. Note that $\tilde{Z}^* \pm 5 \times 10^{-5} \in [z_-, z_+]$ and $\tilde{X}^* \pm 5 \times 10^{-5} \in [x_-, x_+]$, so in view of (17), $\{Z_{n-i}\}_{i=0}^4$ and $\{X_{n-i}\}_{i=0}^4$ belong to these intervals, too. Thus, we find $|A| < 1.3 \times 10^{-8}$.

For the estimate of $|B|$ we return to the initial variables and use (17):

$$|B| \leq \Big(\sum_{i=0}^{n-6} \big| B_i b_i^{2(n+1-i)} \big| + |d_{n+1}^{(0)}| \Big) \Big/ |B_n b_n^2| < 10^{-8}.$$

We also easily find $|C| < (\tilde{Z}^*)^{36}/(\tilde{X}^*)^{42} < (z_+)^{36}/(x_-)^{42} < 3 \times 10^{-9}$. As a result we have $|\epsilon_{n+1}| < 3 \times 10^{-8}$. In a very similar way we estimate

$$\begin{aligned}
\delta_{n+1} &= -\frac{Z_n^2 Z_{n-1}^4 Z_{n-2}^6 Z_{n-3}^8 Z_{n-4}^{10}}{X_n^3 X_{n-1}^5 X_{n-2}^7 X_{n-3}^9 X_{n-4}^{11}} + - \cdots \text{ (to } X_1) + \frac{d_{n+1}^{(1)}}{B_n b_n^3} - \sum_{i=5}^{\infty} (-1)^i \frac{(\tilde{Z}^*)^{i(i+1)}}{(\tilde{X}^*)^{i(i+2)}} \\
&= \Big[-\psi_5(Z_n, \ldots, X_{n-4}) + \frac{(\tilde{Z}^*)^{30}}{(\tilde{X}^*)^{35}} \Big] + \Big[\cdots \Big] - \sum_{i=6}^{\infty} (-1)^i \frac{(\tilde{Z}^*)^{i(i+1)}}{(\tilde{X}^*)^{i(i+2)}} \\
&=: A_1 + B_1 + C_1,
\end{aligned}$$

whence $|\delta_{n+1}| \leq |A_1| + |B_1| + |C_1| < 7 \times 10^{-10} + 10^{-8} + 6 \times 10^{-11} < 2 \times 10^{-8}$.

Thus we obtain $||\overline{\varepsilon}^1|| < 3 \times 10^{-8}$ and hence $||\overline{\varepsilon}_n|| \leq \sum_{i=1}^4 ||\overline{\varepsilon}^i|| < 9 \times 10^{-8} < \varepsilon$. The relation (19) is proved under the corresponding conditions.

Now we will prove the following

Claim. Let conditions (17) are fulfilled with $r = 2 \times 10^{-5}$ and assume that $|Z_{k+i} - \tilde{Z}^*|$, $|X_{k+i} - \tilde{X}^*| \leq r$, $i = 1, \ldots, 6$. Then $||\overline{V}_n - \tilde{V}^*|| \leq r \ \ \forall n \geq k$, i.e. the above inequalities hold for all $i \geq 0$.

First note that from (17) with any positive $r \leq 2 \times 10^{-5}$ they follow all but the first two similar inequalities with $k+1$ in place of k. Indeed, the relations $|Z_{k+1-i} - \tilde{Z}^*|$, $|X_{k+1-i} - \tilde{X}^*| \leq r$, $i = 1, 2, 3$ are contained in (17) and the inequalities $|Z_{k+1-4} - \tilde{Z}^*|$, $|X_{k+1-4} - \tilde{X}^*| \leq 5 \times 10^{-5}$ are obvious consequences.

The last condition follows from $|b_k| = |Y_k b_{k-1}| > \frac{x_-}{z_+}|b_{k-1}| > 1.894|b_{k-1}|$. It remains to estimate for $j = 0, 1$

$$
\begin{aligned}
R^{(j)} &:= \Big(1 + \sum_{i=0}^{k-5} \big|B_i b_i^{2(k+2-i)+j}\big|\Big) \Big/ \big|B_{k+1} b_{k+1}^{2+j}\big| \\
&\le \Big(1 + b_{k-1}^{2+j} \sum_{i=0}^{k-6} \big|B_i b_i^{2(k+1-i)}\big| + \big|B_{k-5} b_{k-5}^{14+j}\big|\Big) \Big/ \big|B_k b_k^2\big| \cdot \Big|\frac{B_k b_k^2}{B_{k+1} b_{k+1}^{2+j}}\Big| \\
&\le b_{k-1}^{2+j} \Big(1 + \sum_{i=0}^{k-6} \big|B_i b_i^{2(k+1-i)}\big| + \Big(\frac{b_{k-5}}{b_{k-1}}\Big)^{2+j} \big|B_{k-5} b_{k-5}^{12}\big|\Big) \Big/ \big|B_k b_k^2\big| \cdot \Big|\frac{1}{Z_{k+1} b_{k+1}^{2+j}}\Big| \\
&\le \frac{1}{\big|Z_{k+1}(Y_{k+1}Y_k)^{2+j}\big|} \Big[10^{-8} + \big(Y_{k-1} ... Y_{k-4}\big)^{-2-j} |\varphi_5(Z_k, ..., X_{k-4})|\Big] \\
&\le \frac{Z_{k+1} Z_k^2}{\big|X_{k+1}^2 X_k^2 (Y_{k+1}Y_k)^j\big|} \big[10^{-8} + (x_-/z_+)^{-8-4j} z_+^{25}/x_-^{30}\big],
\end{aligned}
$$

where we have used $Y_n = X_n/Z_n$ and $Z_{k-i} < z_+$, $X_{k-i} > x_-$ for $i = 0, \ldots, 4$. Now, by (19) we conclude that $||\overline{V}_{k+1} - \tilde{V}^*|| \le ||\tilde{J}||r + \varepsilon < 8.54r + \varepsilon < 2 \times 10^{-4}$, which implies $Z_{k+1} < 0.3086$, $X_{k+1} > 0.5840$ and $Y_{k+1} > 1$. Therefore, in view of $Y_k > 1$, we get $R^{(j)} < 0.2522\big[10^{-8} + z_+^{33}/x_-^{38}\big] < 10^{-8}$, $j = 0, 1$.

From this and the conditions of the claim we conclude that (17) and (19) are fulfilled for $n = k, \ldots, k+6$. Therefore we have

$$
\begin{aligned}
||\overline{V}_{k+7} - \tilde{V}^*|| &= \big\|\tilde{J}.(\overline{V}_{k+6} - \tilde{V}^*) + \overline{\varepsilon}_{k+6}\big\| = \cdots = \Big\|\tilde{J}^7.(\overline{V}_k - \tilde{V}^*) + \sum_{i=0}^{6} \tilde{J}^i \overline{\varepsilon}_{k+6-i}\Big\| \\
&\le \big\|\tilde{J}^7\big\|.r + \sum_{i=0}^{6} \big\|\tilde{J}^i\big\|.\varepsilon\,.
\end{aligned}
$$

We calculated $\|\tilde{J}\| \approx 8.53$, $\|\tilde{J}^2\| \approx 11.13$, $\|\tilde{J}^3\| \approx 11.13$, $\|\tilde{J}^4\| \approx 11.13$, $\|\tilde{J}^5\| \approx 11.13$, $\|\tilde{J}^6\| \approx 2.38$ and $\|\tilde{J}^7\| \approx 0.274$. Thus we obtain

$$
||\overline{V}_{k+7} - \tilde{V}^*|| < 0.275r + 56.5\varepsilon < 1.2 \times 10^{-5} < r\,,
$$

and the claim follows by induction.

To accomplish the proof of Proposition 4 it remains to show that conditions (17) with $r = 10^{-6}$ imply the conditions of the claim. Indeed, it follows from the claim that $b_n/b_{n-1} = Y_n > x_-/z_+ > 1.894$ for $n \ge k$, and hence $|B_n/B_{n-1}| = Z_n b_{n-1}^2 > (z_-) b_{n-1}^2 \to \infty$.

Let (17) be fulfilled with $r = 10^{-6}$ and for a $j \in \{1, \ldots, 6\}$, $\|\overline{V}_{k+i} - \tilde{V}^*\| \le 2 \times 10^{-5}, i = 1, \ldots, j-1$. Then (19) holds for $n = k, \ldots, k+j-1$ and

$$
\begin{aligned}
\|\overline{V}_{k+j} - \tilde{V}^*\| &= \big\|\tilde{J}^j.(\overline{V}_k - \tilde{V}^*) + \tilde{J}^{j-1}.\overline{\varepsilon}_k + \cdots + \tilde{J}^0.\overline{\varepsilon}_{k+j-1}\big\| \\
&\le 11.14\, r + 54.1\, \varepsilon < 2 \times 10^{-5}.
\end{aligned}
$$

This observation inductively implies the conditions of the claim and completes the proof of the stability of (Z^*, X^*), i.e. the proof of Proposition 4. $\square$

Proof of Theorem 1. We apply Proposition 4 to the series a) for $l(x)$ with $B_n = A_n$, $b_n = a_n$ and $d_n^{(j)} = \frac{1}{2n+1+j}$, $j = 0, 1$. It is calculated that for $k = 13$ the conditions of the proposition are fulfilled. Namely,
$\{Z_n - \tilde{Z}^*\}_{n=9}^{13} \approx \{1 \times 10^{-6}, 9 \times 10^{-8}, -6 \times 10^{-8}, -1 \times 10^{-8}, -1 \times 10^{-9}\}$;
$\{X_n - \tilde{X}^*\}_{n=9}^{13} \approx \{2 \times 10^{-6}, 7 \times 10^{-8}, -9 \times 10^{-8}, -1 \times 10^{-8}, 7 \times 10^{-9}\}$;
$\left(1 + \sum_{i=0}^{k-6} \left|A_i a_i^{2(k+1-i)+j}\right|\right) \Big/ \left|A_k a_k^{2+j}\right| \approx 3 \times 10^{-9}, 6 \times 10^{-11}$ for $j = 0, 1$;
$a_{k-1} = \max_{i \le k-1} |a_i| \approx 1744.92 \ge 1$.

Then, by Proposition 4, $|A_n|$ tends to ∞ faster than any geometrical series, while a_n in the denominator behave as $C.(Y^*)^n$. Therefore, the common term in the sum of a) does not tend to 0 (unless for $x = 0$) and the series diverges.

A very similar argument holds for the series b). Now, $d_n^{(j)} = \frac{1}{4n+1+2j}$, $j = 0, 1$ and again for $k = 13$ the conditions of Proposition 4 are fulfilled:
$\{Z_n - \tilde{Z}^*\}_{n=9}^{13} \approx \{1 \times 10^{-6}, 5 \times 10^{-7}, -3 \times 10^{-8}, -3 \times 10^{-8}, -5 \times 10^{-9}\}$;
$\{X_n - \tilde{X}^*\}_{n=9}^{13} \approx \{3 \times 10^{-6}, 7 \times 10^{-7}, -7 \times 10^{-8}, -4 \times 10^{-8}, 2 \times 10^{-9}\}$;
$\left(1 + \sum_{i=0}^{k-6} \left|B_i b_i^{2(k+1-i)+j}\right|\right) \Big/ \left|B_k b_k^{2+j}\right| \approx 3 \times 10^{-9}, 6 \times 10^{-11}$ for $j = 0, 1$;
$b_{k-1} = \max_{i \le k-1} |b_i| \approx 1399.65 \ge 1$.

Then, Proposition 4 implies the divergence of b) for $z \neq 0$. Theorem 1 is proved. $\square$

5. BOUNDS FOR THE RATE OF CONVERGENCE OF (2) AND (3).

We first note an useful formula connecting the coefficients in the representation

$$\frac{A_0}{1-\alpha_0 x} + \frac{A_1 x}{(1-\alpha_0 x)(1-\alpha_1 x)} + \frac{A_2 x^2}{(1-\alpha_0 x)(1-\alpha_1 x)(1-\alpha_2 x)} + \cdots \approx \sum_{n=0}^{\infty} a_n x^n.$$

Namely,

$$\begin{aligned}
&A_0 = a_0; \quad A_1 = a_1 - \alpha_0 a_0; \\
&A_2 = a_2 - (\alpha_0 + \alpha_1) a_1 + (\alpha_0 \alpha_1) a_0; \\
&A_3 = a_3 - (\alpha_0 + \alpha_1 + \alpha_2) a_2 + (\alpha_0 \alpha_1 + \alpha_1 \alpha_2 + \alpha_2 \alpha_0) a_1 - (\alpha_0 \alpha_1 \alpha_2) a_0; \\
&A_4 = a_4 - \sigma_1(\alpha_0, ..., \alpha_3) a_3 + \sigma_2(\alpha_0, ..., \alpha_3) a_2 - \sigma_3(\alpha_0, ..., \alpha_3) a_1 + \sigma_4(\alpha_0, ..., \alpha_3) a_0; \\
&\quad \vdots
\end{aligned} \tag{20}$$

where $\sigma_k(\alpha_0, \ldots, \alpha_n) = \sum_{0 \le i_1 < \cdots < i_k \le n} \alpha_{i_1} \ldots \alpha_{i_k}$. Formulas (20) easily follow by induction. Indeed, the relations for A_0 and A_1 are easily verified. Let the formula holds for A_k with a fixed $k \geq 1$ and arbitrary parameters $\{\alpha_i\}$ and $\{a_i\}$. Then, removing the denominator $1-\alpha_0 x$, subtracting A_0 and dividing by x we obtain

$$\frac{A_1}{(1-\alpha_1 x)} + \frac{A_2 x}{(1-\alpha_1 x)(1-\alpha_2 x)} + \cdots \approx \sum_{n=0}^{\infty} a_{n+1} x^n - \alpha_0 \sum_{n=0}^{\infty} a_n x^n.$$

So, by the induction and the linearity, for the coefficient of $x^k/\prod_{i=1}^{k+1}(1-\alpha_i x)$ we find

$$\begin{aligned}
A_{k+1} =& \big[a_{k+1} - \sigma_1(\alpha_1,...,\alpha_k)a_k + \sigma_2(\alpha_1,...,\alpha_k)a_{k-1} - \cdots + (-1)^k \sigma_k(\alpha_1,...,\alpha_k)a_1\big] \\
&- \alpha_0\big[a_k - \sigma_1(\alpha_1,...,\alpha_k)a_{k-1} + \cdots + (-1)^k \sigma_k(\alpha_1,...,\alpha_k)a_0\big] \\
=& a_{k+1} - \sigma_1(\alpha_0,...,\alpha_k)a_k + \big(\sigma_2(\alpha_1,...,\alpha_k) + \alpha_0\sigma_1(\alpha_1,...,\alpha_k)\big)a_{k-1} - + \cdots \\
&+ (-1)^k\big(\sigma_k(\alpha_1,...,\alpha_k) + \alpha_0\sigma_{k-1}(\alpha_1,...,\alpha_k)\big)a_1 + (-1)^{k+1}\alpha_0\alpha_1...\alpha_k \cdot a_0\,,
\end{aligned}$$

as for $k=1$ the middle terms in the brackets do not appear. For $k \geq 2$ the induction step follows by the properties of the combinatorial sums $\{\sigma_i\}$.

In the particular case $a_0 = \alpha_0 = 0$ and $a_n = \frac{1}{n}$, $n \geq 1$, we arrive at the formula

$$l(x) \approx \frac{C_1 x}{1-\gamma_1 x} + \frac{C_2 x^2}{(1-\gamma_1 x)(1-\gamma_2 x)} + \frac{C_3 x^3}{(1-\gamma_1 x)(1-\gamma_2 x)(1-\gamma_3 x)} + \cdots\,,$$

where

$$\begin{aligned}
C_k &= \frac{1}{k} - \sigma_1(\gamma_1,...,\gamma_{k-1})\frac{1}{k-1} + \sigma_2(\gamma_1,...,\gamma_{k-1})\frac{1}{k-2} - \cdots + (-1)^{k-1}(\gamma_1\cdots\gamma_{k-1}) \\
&= \int_0^1 (x-\gamma_1)...(x-\gamma_{k-1})\,d\,x. \qquad (21)
\end{aligned}$$

Let us consider some concrete representations of $l(x)$ with periodic $\{\gamma_i\}$. As was mentioned before, the choice $\{\gamma_i\}_1^\infty = \{\frac{1}{2}\}_1^\infty$ leads to the series (5). Let now $\{\gamma_i\}_1^\infty = \{0, a, b, a, b, ...\}$. We take $\gamma_1 = 0$ in order to write the series in the form

$$l(x) = D_0 x + (B_1 + D_1 x)u + (B_2 + D_2 x)u^2 + (B_3 + D_3 x)u^3 + \cdots\,, \qquad (22)$$

where $u = \dfrac{x^2}{(1-ax)(1-bx)}$. Then $D_0 = 1$ while for $n \geq 1$, from (21) and

$$(B_n + D_n x)u^n = \frac{B_n x^{2n}}{(1-0.x)(1-ax)^n(1-bx)^{n-1}} + \frac{(bB_n + D_n)x^{2n+1}}{(1-0.x)(1-ax)^n(1-bx)^n},$$

it follows that $B_n = \int_0^1 x\big[(x-a)(x-b)\big]^{n-1} d\,x$, $D_n + bB_n = \int_0^1 x(x-a)^n(x-b)^{n-1} d\,x$, hence $D_n = \int_0^1 x^2\big[(x-a)(x-b)\big]^{n-1} d\,x - sB_n$, where $s = a+b$.

Introducing $A_n = \int_0^1 \big[(x-a)(x-b)\big]^{n-1} d\,x$, $C_n = \int_0^1 x^2\big[(x-a)(x-b)\big]^{n-1} d\,x$ and $a' = 1-a$, $b' = 1-b$, one can easily verify the recurrence relations

$$\begin{aligned}
B_n &= \frac{(a'b')^n - (ab)^n}{2n} + \frac{s}{2}A_n; \qquad & C_n &= \frac{(a'b')^n + (n+1)sB_n - abA_n}{2n+1}; \\
D_n &= C_n - sB_n; & A_{n+1} &= D_n + abA_n.
\end{aligned}$$

To minimize the asymptotics of the coefficients in (22) we choose $(x-a)(x-b) = x^2 - x + \frac{1}{8}$, which is the Chebyshev polynomial of the first kind $T_2^*(x)$, transformed to the interval $[0,1]$. Recall that the polynomials $T_k^*(x) = x^k + \cdots$ associated with the interval $[\alpha, \beta]$ provide the minimal uniform norm on $[\alpha, \beta]$ amongst all polynomials of the form $x^k + \sum_{i=0}^{k-1} a_i x^i$ and the value of this minimal norm is $\frac{(\beta-\alpha)^k}{2^{2k-1}}$ ([4, Ch.2.2.3]). In the our case, $\|T_2^*\|_{C[0,1]} = -T_2^*(1/2) = \frac{1}{8}$ which yields that the above integrals are asymptotic to $\frac{const}{\sqrt{n}}\Big(-\frac{1}{8}\Big)^n$ for $n \to \infty$. For this choice we have $s = a + b = 1$ and the recurrence relations simplify to

$$A_{n+1} = \frac{1}{2n+1}\Big[\frac{1}{8^n} - \frac{n}{4}A_n\Big]; \quad B_n = \frac{1}{2}A_n; \quad D_n = A_{n+1} - \frac{1}{8}A_n.$$

It is convenient to substitute $A_n = \Big(\frac{1}{8}\Big)^{n-1}\alpha_n$, where $\alpha_{n+1} = \dfrac{1 - 2n\alpha_n}{2n+1}$. Then, the rate of convergence of this special case of (22), considered as a series of type (3), i.e. with each summand counted twice, is like a geometrical series with ratio $\frac{\sqrt{u}}{2\sqrt{2}} \sim \frac{x}{2\sqrt{2}}$ $(x \to 0)$. In addition we can rewrite the series in a lacunary form. Namely, by the recurrence formulas and $D_0 = A_1 = 1$ we get

$$\begin{aligned} l(x) =&\ A_1 x + \Big(\frac{1}{2}A_1 + \big(A_2 - \frac{1}{8}A_1\big)x\Big)u + \Big(\frac{1}{2}A_2 + \big(A_3 - \frac{1}{8}A_2\big)x\Big)u^2 + \cdots \\ =&\ \Big[x + \Big(\frac{1}{2} - \frac{x}{8}\Big)u\Big]\Big[A_1 + A_2 u + A_3 u^2 + \cdots\Big], \end{aligned} \tag{23}$$

where A_n satisfies the above formulas and $u = \frac{x^2}{1-x+x^2/8}$. (The second factor is lacunary, considered as a series of the form (3).)

Remark 3. Although the series (23) converges faster than (5), and is lacunary as well, it is still less effective. This is because the coefficients are more complicated. Indeed, let us count only multiplications and divisions as the most costly arithmetic operations with equal cost. Then every next term in (5) needs two operations ($z^{2n+1} = z^{2n-1} \times z^2$ and $z^{2n+1}/(2n+1)$), while every next term in (23) needs three operations ($\alpha_{n+1} = (1+\alpha_n)/(2n+1) - \alpha_n$, $(u/8)^n = (u/8)^{n-1} \times (u/8)$ and $\alpha_{n+1} \times (u/8)^n$). Actually, even the example below hardly improves (5).

Remark 4. We see that the first factor in (23) vanishes for $x = 0$ and $x = 2$. In fact, $l(0) = 0$ but $l(2) = \log(-1) \neq 0$. Recall our adoption that when the region of validity of some identity is not specified, it is certain neighborhood of 0. In particular, (23) converges for $|x| < 1$ and represents $l(x)$ in the open unit disc. On the other hand, a continuation of (23) for x outside the unit disc is questionable because of $l(1) = \infty$.

Consider now "periodic" representations

$$l(x) = b_0 x + c_0 x^2 + \sum_{n=1}^{\infty}(a_n + b_n x + c_n x^2)v^n, \quad v = \frac{x^3}{(1-ax)(1-bx)(1-cx)},$$

i.e. of the form (3) with $\{\gamma_i\}_1^\infty = \{0, 0, a, b, c, a, b, c, ...\}$. Transforming (3) in the above form, in view of (21), we find the following integral formulas for $n \geq 1$:

$$\begin{aligned}
a_n &= \int_0^1 t^2 \big[(t-a)(t-b)(t-c)\big]^{n-1}\, dt; \\
b_n &= \int_0^1 t^2 (t-a-b-c) \big[(t-a)(t-b)(t-c)\big]^{n-1}\, dt; \\
c_n &= \int_0^1 t^2 \big(t^2 - (a+b+c)t + ab + bc + ca\big) \big[(t-a)(t-b)(t-c)\big]^{n-1}\, dt.
\end{aligned}$$

For simplicity let us assume that the points a, b and c are symmetrically placed in $[0, 1]$, or more precisely let $a + b = 1$ and $c = 1/2$. Then, with the notations $I_n := \int_0^1 [P(t)]^n dt$ and $J_n := \int_0^1 t[P(t)]^n dt$ where $P(t) = (t-a)(t-b)(t-c)$, we easily get $I_{2m-1} = 0$ and $J_{2m} = \frac{1}{2} I_{2m}$. Next, using $P(t) = \big(\frac{1}{3}P'(t) + \frac{4ab-1}{6}\big)\big(t - \frac{1}{2}\big)$ we calculate

$$\begin{aligned}
I_n &= \frac{n/2}{3n+1}\Big[(4ab-1)\Big(J_{n-1} - \frac{1}{2} I_{n-1}\Big) + \frac{1-(-1)^n}{n}\Big(\frac{ab}{2}\Big)^n\Big]; \\
J_n &= \frac{n/2}{3n+2}\Big[\frac{1}{n} I_n + (2ab - \tfrac{1}{2})\Big(J_{n-1} - \frac{2ab+1}{3} I_{n-1}\Big) \\
&\qquad + \Big(\frac{ab}{2}\Big)^n \Big(\frac{1}{n} + (4ab-1)\frac{1-(-1)^n}{3n}\Big)\Big]; \\
a_n &= J_{n-1} - \frac{2ab+1}{6} I_{n-1} + \frac{1-(-1)^n}{3n}\Big(\frac{ab}{2}\Big)^n; \\
b_n &= I_n - \big(ab + \frac{1}{2}\big) J_{n-1} + \frac{ab}{2} I_{n-1}; \\
c_n &= J_n + \frac{ab}{2} J_{n-1}.
\end{aligned}$$

In particular, a nice formula is obtained if we take $a = 0$ and $b = 1$. Then

$$J_n = \begin{cases} -\frac{1}{12}\frac{n}{3n+2} J_{n-1} & \text{for odd } \; n > 0 \\ -\frac{1}{4}\frac{n}{3n+1} J_{n-1} & \text{for even } \; n > 0\,, \end{cases}$$

$$a_n + b_n x + c_n x^2 = \begin{cases} \big(\frac{2}{3} - \frac{x}{2}\big) J_{n-1} + x^2 J_n & \text{for odd } \; n \\ \big(1 - \frac{x}{2}\big) J_{n-1} + (x^2 + 2x) J_n & \text{for even } \; n\,. \end{cases}$$

The starting value is $J_0 = \frac{1}{2}$ and even the second formula holds for $n = 0$ with $J_{-1} := 0$. Therefore, with $v = \frac{x^3}{(1-x/2)(1-x)}$,

$$\begin{aligned}
l(x) \;=\; & \Big(x + \frac{x^2}{2} + \Big(\frac{1}{3} - \frac{x}{4}\Big)v\Big)\Big[1 + \frac{1}{48}\frac{2!v^2}{5.7} + \frac{1}{48^2}\frac{4!v^4}{5.7.11.13} + \frac{1}{48^3}\frac{6!v^6}{5.7.11.13.17.19} + \cdots\Big] \\
& - \frac{1}{24}\Big(x^2 + \Big(1 - \frac{x}{2}\Big)v\Big)\Big[\frac{1!v}{5} + \frac{1}{48}\frac{3!v^3}{5.7.11} + \frac{1}{48^2}\frac{5!v^5}{5.7.11.13.17} + \cdots\Big].
\end{aligned} \tag{24}$$

The both parts of (24), considered together and as a series of type (3), converge like a geometric series with ratio $\sqrt[6]{\frac{1}{48\cdot 9}}x \approx \frac{x}{2.75}$ $(x \to 0)$.

Proof of Theorem 2. To prove a), we denote by $0 < x_1 < x_2 < \cdots < x_k < 1$ the zeros of the Chebyshev polynomial $T_k^*(x)$ for the interval $[0,1]$ and take the periodic sequence $\{\gamma_i\}_1^\infty = \{x_1, \ldots, x_k, x_1, \ldots, x_k, \ldots\}$. Then, by (21) we have

$$|C_{n+1}| \le \int_0^1 |T_k^*(t)|^m \, |(t-\gamma_{km+1})...(t-\gamma_n)| d\,t \le \left(2^{-2k+1}\right)^m < 2^{(-2k+1)(n/k-1)},$$

where $n = k\,m + r,\quad r \in \{0, \ldots, k-1\}$. Hence,

$$\limsup_{n\to\infty} |C_{n+1}|^{1/n} \le 4^{-1+1/2k} =: q(k).$$

Therefore, with a fixed $\varepsilon \in (0,1]$, the inequality $q(k) < \frac{1}{4-\varepsilon}$ holds true, provided k is sufficiently large. With such a k, the sequence $\{\gamma_i\}$ above satisfies a).

To prove b) we shall exploit some properties of the Legendre polynomials $P_n(x) = \frac{1}{2^n n!}[(x^2-1)^n]^{(n)}$. Let $p_n(x) = P_n(2x-1)$ be the normalized Legendre polynomials for the interval $[0,1]$. It follows from $\int_{-1}^1 P_n(x)Q_m(x)d\,x = 0$ for every polynomial $Q_m(x)$ of degree $m < n$ that $\int_0^1 p_n(x)q_m(x)d\,x = 0$ provided $\deg(q_m) = m < n$. In addition, $\int_{-1}^1 P_n^2(x)d\,x = \frac{2}{2n+1}$ implies $\int_0^1 p_n^2(x)d\,x = \frac{1}{2n+1}$.

Now consider a representation of $l(x)$ in the form (3) with $\gamma_i \in [0,1]$. By (21) $C_{n+1} = \int_0^1 \o_n(t)d\,t$, where $\o_n(x) := \prod_{i=1}^n (x-\gamma_i)$. Given a fixed $n \in \mathbb{N}$, let us represent $p_n(x)$ by the Newton interpolation formula at the points $\{\gamma_i\}_{i=n+1}^{2n+1}$, namely

$$\begin{aligned}
p_n(x) &= \sum_{k=0}^n p_n[\gamma_{n+1}, ..., \gamma_{n+k+1}](x-\gamma_{n+1})...(x-\gamma_{n+k}) \\
&= \sum_{k=0}^n \frac{p_n^{(k)}(\eta_k)}{k!}(x-\gamma_{n+1})\ldots(x-\gamma_{n+k}),
\end{aligned}$$

where $\eta_k \in [0,1],\ k = 0, \ldots, n$. Now we will use the relation $P_n(x) = P_n^{\left(\frac{1}{2}\right)}(x)$ and the following properties (see [7, Ch.4.7,7.33]) of the ultraspherical polynomials $P_n^{(\lambda)}(x)$ (note that here λ represents a parameter, and not derivative order): $\frac{d}{dx}\left\{P_m^{(\lambda)}(x)\right\} = 2\lambda P_{m-1}^{(\lambda+1)}(x)$; $P_m^{(\lambda)}(1) = \binom{m+2\lambda-1}{m}$ and $\max\limits_{-1\le x\le 1}|P_m^{(\lambda)}(x)| = P_m^{(\lambda)}(1)$, for $\lambda > 0$. Then, with $\lambda = k + \frac{1}{2}$, we have $P_n^{(k)}(x) = (2k-1)!!P_{n-k}^{(\lambda)}(x)$ and hence, for $k = 0, \ldots, n$,

$$|p_n^{(k)}(\eta_k)| = 2^k |P_n^{(k)}(2\eta_k - 1)| \le 2^k \|P_n^{(k)}\|_{C[-1,1]} = 2^k P_n^{(k)}(1) = \frac{(n+k)!}{k!(n-k)!}.$$

In particular, for $k = n$ we find that the leading coefficient of $p_n(x)$ is $q_n = \binom{2n}{n}$. Thus, from the above representation of $p_n(x)$ and (21) we obtain

$$I_n := \int_0^1 \o_n(x)p_n(x)d\,x = \sum_{k=0}^{n} \frac{p_n^{(k)}(\eta_k)}{k!} C_{n+k+1}. \tag{25}$$

On the other hand

$$I_n = q_n^{-1} \int_0^1 \big((q_n \o_n(x) - p_n(x)) + p_n(x)\big)p_n(x)d\,x = q_n^{-1} \int_0^1 p_n^2(x)d\,x = \frac{(n!)^2}{(2n+1)!}.$$

Therefore, for at least one summand in (25) we have that

$$\begin{aligned}\frac{(n!)^2}{(n+1)(2n+1)!} &\le \frac{p_n^{(k)}(\eta_k)}{k!} C_{n+k+1} \\ &\le \frac{(n+k)!}{(k!)^2(n-k)!}|C_{n+k+1}| = \tbinom{n+k}{2k}\tbinom{2k}{k}|C_{n+k+1}| < 2^{n+3k}|C_{n+k+1}|,\end{aligned}$$

i.e. $|C_{n+k+1}| > 2^{-n-3k} \cdot \frac{(n!)^2}{(n+1)(2n+1)!} = \frac{2^{-n-3k}}{(n+1)(2n+1)\binom{2n}{n}} > \frac{1}{(n+1)(2n+1)}\left(\frac{1}{8}\right)^{n+k}$. As a consequence, the inequalities $|C_j| \le \frac{M}{(8+\varepsilon)^j}$ $\forall j \in \mathbb{N}$ can not hold true for any positive M and ε. Hence part b) and the theorem are proved. $\square$

Proof of Theorem 3. First we shall prove the assertion for $\{C_n\}$. We can apply the same reasoning as in the proof of part b) of Theorem 2 to the estimation

$$\frac{(n!)^2}{(n+1)(2n+1)!} \le \frac{p_n^{(k)}(\eta_k)}{k!} C_{n+k+1}, \tag{26}$$

for some $k \in \{0, \dots, n\}$, but now the restriction is $|\eta_k| \le 1$. So, we need of an upper bound for $|p_n^{(k)}(\eta_k)|$. In view of the monotonicity of $|P_n^{(k)}(x)|$ for $|x| \ge 1$ we have

$$\begin{aligned}|p_n^{(k)}(\eta_k)| &= 2^k|P_n^{(k)}(2\eta_k - 1)| \le 2^k|P_n^{(k)}(-3)| = 2^k|P_n^{(k)}(3)| \\ &= 2^k Q_{n,k}(3^2 - x_{1,k}^2)(3^2 - x_{2,k}^2)\dots \le 2^k Q_{n,k}\, 3^{n-k},\end{aligned}$$

where $Q_{n,k}$ and $\{\pm x_{i,k}\}$ are the leading coefficient and the zeros of $P_n^{(k)}$, respectively. From the definition of $P_n(x)$ (by the Rodrigues' formula) we find $Q_{n,k} = \frac{(2n)!}{2^n n!(n-k)!}$ from where $|p_n^{(k)}(\eta_k)| \le \frac{(2n)!}{n!(n-k)!}\left(\frac{3}{2}\right)^{n-k}$. Hence, taking modulus in (26), we can write

$$\begin{aligned}(n+1)(2n+1)|C_{n+k+1}| &\ge \frac{(n!)^2}{(2n)!} \cdot \frac{k!}{|p_n^{(k)}(\eta_k)|} \ge \left[\binom{2n}{n}^2 \binom{n}{k}\left(\frac{3}{2}\right)^{n-k}\right]^{-1} \\ &> \left[4^{2n}\binom{n}{k}1^k\left(\frac{3a}{2}\right)^{n-k}a^{k-n}\right]^{-1} > \left[4^{2n}\left(1 + \frac{3a}{2}\right)^n a^{k-n}\right]^{-1} = \left[\left(\frac{16}{a} + 24\right)^n a^k\right]^{-1}.\end{aligned}$$

Now, if we choose $a = 12 + \sqrt{160} < 25$, that is, the positive root of the equation $z = 16/z + 24$, then for any fixed $M > 0$ we obtain $|C_{n+k+1}| > \frac{1}{(n+1)(2n+1)} a^{-n-k} > M \cdot 25^{-(n+k+1)}$, provided n is sufficiently large. The assertion for $\{C_n\}$ is proved.

In order to prove the impossibility of the bounds for $\{A_n\}$ in the theorem, let us assume that for some $M > 0$, $q < 1$ and $\{\alpha_n\}$ such that $|\alpha_n| \le 1$, the estimates $|A_n| \le Mq^n$, $n = 1, 2, \dots$ hold true. Then for $|x| < 1$ we have

$$l(x^2) = l(x) + l(-x) = \sum_{m=1}^{\infty} \frac{2A_{2m-1}\alpha_{2m-1}x^{2m}}{1 - \alpha_{2m-1}^2 x^2} + \frac{2A_{2m}x^{2m}}{1 - \alpha_{2m}^2 x^2}.$$

That is, a representation $l(u) = \displaystyle\sum_{n=1}^{\infty} \frac{B_n u^{\lfloor (n+1)/2 \rfloor}}{1 - \beta_n u}$ holds true for $|u| < 1$, where $\beta_n \in [0, 1]$ and $|B_n| \le 2Mq^n$. We shall show that this series can be written in the form (3) with parameters $\{\gamma_i\} = \{\beta_1, \beta_2, 0, \beta_3, \beta_4, 0, \beta_5, \beta_6, \dots\}$. This will follow from the possibility of the representations

$$\frac{u^m}{1 - \beta_n u} = \sum_{j=1}^{k} \frac{a_{j,n} u^j}{(1 - \gamma_1 u) \dots (1 - \gamma_j u)} =: \sum_{j=1}^{k} a_{j,n} c_j(u), \tag{27}$$

where $m = \lfloor \frac{n+1}{2} \rfloor$ and $k = n + m - 1 = \lfloor \frac{3n-1}{2} \rfloor$, i.e $\gamma_k = \beta_n$. Note that $\{\gamma_i\}_1^k$ contains $m - 1$ zeros. Then, with $v = 1/u$ we have $c_j(u) = \frac{1}{(v-\gamma_1)\dots(v-\gamma_j)}$ and (for $v \ne 0, \gamma_1, \dots, \gamma_k$) equality (27) is equivalent to

$$g_{n-1}(v) := (v - \beta_1) \dots (v - \beta_{n-1}) = \sum_{i=0}^{k-1} a_{k-i,n} (v - \gamma_k) \dots (v - \gamma_{k-i+1}),$$

where the indices of $\{\gamma_j\}$ decrease, so the first product in the sum is assumed equal to 1. Thus, we have a representation of $g_{n-1}(x)$ by the Newton interpolating formula at the points $\gamma_k, \dots, \gamma_1$, hence ($k \ge n$) the representation exists and $a_{j,n} = g_{n-1}[\gamma_k, \dots, \gamma_j]$, $j = 1, \dots, k$. As a consequence, $a_{k-i,n} = \frac{g_{n-1}^{(i)}(\xi_i)}{i!}$, where $\xi_i \in [0, 1]$ since $\{\gamma_j\} \subset [0, 1]$. The last equality implies that $a_{1,n} = \dots = a_{m-1,n} = 0$ and

$$|a_{k-i,n}| = \binom{n-1}{i} |(\xi_i - x_1^{(i)}) \dots (\xi_i - x_{n-1-i}^{(i)})| \le \binom{n-1}{i}, \quad i = 0, \dots, n-1,$$

where $\{x_j^{(i)}\}$ are the zeros of $g_{n-1}^{(i)}(x)$. Using these estimates and (27) we obtain

$$\begin{aligned} l(u) &= \sum_{n=1}^{\infty} \frac{B_n u^m}{1 - \beta_n u} = \sum_{n=1}^{\infty} B_n \sum_{\frac{n}{2} \le j \le \frac{3n-1}{2}} a_{j,n} c_j(u) = \sum_{j=1}^{\infty} c_j(u) \sum_{\frac{2j+1}{3} \le n \le 2j} B_n a_{j,n} \\ &=: \sum_{j=1}^{\infty} C_j c_j(u). \end{aligned}$$

Thus we have written $l(u)$ in the form (3) with coefficients that satisfy

$$|C_j| \le \sum_{\frac{2j+1}{3} \le n \le 2j} 2Mq^n \binom{n-1}{k-j} \le 2M \sum_{\frac{2j+1}{3} \le n \le 2j} q^n \binom{2j-1}{\lfloor \frac{3n-1}{2} \rfloor - j}.$$

The numbers $l_n(j) := \lfloor \frac{3n-1}{2} \rfloor - j,\ n = \lceil \frac{2j+1}{3} \rceil, \ldots, 2j$ belong to $\{0, \ldots, 2j-1\}$ and are distinct. So, in view of $\frac{3n}{2} > \lfloor \frac{3n-1}{2} \rfloor = l_n(j) + j$,

$$|C_j| < 2M \sum_{l=0}^{2j-1} q^{\frac{2}{3}(l+j)} \binom{2j-1}{l} = 2Mq^{\frac{2}{3}j} \big[1 + q^{2/3}\big]^{2j-1} = M_1 \big[q^{1/3} + q\big]^{2j}.$$

Therefore, the assumption $q \le \frac{1}{31}$ leads to $|C_j| < M_1 \left(\frac{1}{8}\right)^j$ which is a contradiction to Theorem 2 since $\gamma_j \in [0,1]$. Theorem 3 is proved. $\square$

Now we shall make a comparison between the form (3) and the method of continued fractions (see e.g. [1, Ch.4] for the used results). The similarity of the two approaches is obvious – in both cases the n-th partial sum of the Maclaurin series is recovered. We mean the usual representation of a function by a continued fraction

$$f(z) = b_0 + \frac{a_1 z}{b_1 +}\ \frac{a_2 z}{b_2 +}\ \frac{a_3 z}{b_3 +}\ \cdots =: b_0 + \mathbf{K}_{i=1}^{\infty}(a_i z / b_i). \tag{28}$$

But there are also some essential differences. Only seemingly the form (28) depends on two sequences $\{a_i\}$ and $\{b_i\}$. In fact, a nonsingular continued fraction (28), i.e. with $a_i, b_i \neq 0,\ i \ge 1$, elementary can be transformed into an equivalent form, say with $a_i = 1$ or with $b_i = 1$. For example, the fraction

$$\log(1+z) = \frac{z}{1+}\ \frac{1^2 z}{2+}\ \frac{1^2 z}{3+}\ \frac{2^2 z}{4+}\ \frac{2^2 z}{5+}\ \cdots\ + \frac{n^2 z}{2n+}\ \frac{n^2 z}{2n+1+}\ \cdots \tag{29}$$

is transformed (by dividing the numerator and the denominator of the $2n$-th and $2n+1$-th terms to $n\sqrt{z}$) into

$$\begin{aligned} log(1+z) \quad = \quad & \frac{\sqrt{z}}{(1/\sqrt{z})+}\ \frac{1}{(2/\sqrt{z})+}\ \frac{1}{(3/1\sqrt{z})+}\ \frac{2/1}{(2/\sqrt{z})+}\ \frac{1}{(5/2\sqrt{z})+}\ \cdots \\ & + \frac{n/(n-1)}{(2/\sqrt{z})+}\ \frac{1}{((2n+1)/n\sqrt{z})+}\ \cdots \end{aligned}$$

This form has the advantage to (29) that it is close to a continued fraction $F = \mathbf{K}_{i=1}^{\infty}(1/b_i)$ (with unit numerators). The convergence of such a fraction is very easy to realize in view of the Seidel's theorem which states that when the elements $\{b_i\}_1^\infty$ are positive, then F is convergent iff the series $\sum_1^\infty b_i$ is divergent. Moreover, for "relatively large" elements (say $|b_i| \ge 3, i \ge n_0$) the fraction converges approximately like $[b_1 b_2 \ldots b_n]^{-2}$. In the case of $\log(1+z)$, and equivalently of $l(x)$, this rule gives an approximate rate of convergence like $[(2/\sqrt{z})^n]^{-2} = (z/4)^n\ (z \to 0)$.

The above arguments can be done precise using the formulas

$$\mathbf{K}_{i=1}^{\infty}(a_i/b_i) = \frac{a_1}{b_1} + \sum_{i=2}^{\infty} \frac{(-1)^{i-1}}{B_i B_{i-1}} \prod_{j=1}^{i} a_j,$$

where $A_n/B_n = b_0 + \mathbf{K}_{i=1}^{n}(a_i/b_i)$ is the n-th convergent of the fraction, and

$$\begin{aligned} A_i &= b_i A_{i-1} + a_i A_{i-2}, \quad A_0 = b_0,\ A_{-1} = 1; \\ B_i &= b_i B_{i-1} + a_i B_{i-2}, \quad B_0 = 1,\ B_{-1} = 0. \end{aligned}$$

In particular, for a fraction with $a_i = 1$ the remainder is

$$R_n := \mathbf{K}_{i=1}^{\infty}(1/b_i) - \mathbf{K}_{i=1}^{n}(1/b_i) = \sum_{i=n+1}^{\infty} \frac{(-1)^{i-1}}{B_i B_{i-1}}.$$

Then, for "relatively large" $|b_i|$, the relation $B_i = b_i B_{i-1} + B_{i-2}$ usually implies $B_i \approx b_i B_{i-1}$ and $B_i \to \infty$ for $i \to \infty$, which in turn yields the approximate rule $|R_n| \approx 1/|B_n B_{n+1}| \approx B_n^{-2}$. It has to be mentioned however, that there are some special cases for the data $\{b_i\}$ when the principal asymptotic behaviour of $\{B_i\}$ as a solution of the above three term recurrence relation is suppressed and the magnitude of the sequence is not the usual one. This corresponds to a fraction of value $1/0 = \infty$ and, in our case of interest, for $l(x)$ with $|x| < 1$, such situations do not appear.

Now, the question is which is the right correspondence for comparing the two methods for accelerating power series? We argue that the most natural way is to compare the n-th partial sum of (3) with the n-th convergent of (28). The calculation of both approximations can be organized in different ways, say backward, and the formal counting of the the cost of arithmetic operations then gives the same result ($2n$). Indeed, the coefficients in the continued fraction for $l(x)$ are much simpler, but (as we have seen) taking $\{\gamma_i\}$ at the zeros of $T_k(x)$ and grouping summands we obtain rational parameters in the series, too.

Let us summarize the above comments. Both methods transform a series with rate of convergence like z^n into a series (sequence) converging approximately as $(z/4)^n$. This accelerating factor $(1/4)^n$ appears often in the continued fraction expansions, for example in

$$L(z) = \operatorname{arcth}(z) = \frac{z}{1-}\ \frac{1^2 z^2}{3-}\ \frac{2^2 z^2}{5-}\ \cdots\ -\frac{n^2 z^2}{2n+1-}\ \cdots\ .$$

Thus, in many cases the both methods have approximately the same efficiency.

6. ACCELERATION OF SERIES FOR OTHER FUNCTIONS

First we consider the function

$$f = f_\alpha(x) := \frac{1}{\alpha} + \frac{x}{1+\alpha} + \frac{x^2}{2+\alpha} + \frac{x^3}{3+\alpha} + \cdots,$$

which contains $l(x)$ and $L(\sqrt{x})$ as particular cases. We describe some transformations of f_α allowing its effective computation. Let us change the variable by $t = \frac{x}{1-x/2}$. An explicit formula for the coefficients in $f = \sum_{n=0}^\infty a_n t^n =: S_0(t)$ can be written using the Euler transform, but it is not convenient for computation and estimation of $\{a_n\}$. More important is the recursive rule, which follows from the differential equation

$$\frac{df}{dt} = -\frac{2\alpha}{t(2+t)} f + \frac{1}{t(1-t/2)},$$

a consequence of $f = \frac{1}{x^\alpha}\int_0^x \frac{z^{\alpha-1}}{1-z}dz = \frac{(1+t/2)^\alpha}{t^\alpha}\int_0^{\frac{t}{1+t/2}} \frac{z^{\alpha-1}}{1-z}dz$. We have

$$a_{n+1} = \Big(\frac{1}{2^n} - \frac{n}{2}a_n\Big)\Big/(n+1+\alpha), \quad a_0 = \frac{1}{\alpha}. \tag{30}$$

When $\alpha > 0$, it is easily seen from (30) that $a_n \in (0, 2^{1-n})$, $n > 0$, hence the transformation gives an acceleration of $f_\alpha(x)$, for $x \to 0$, like 2^{-n}.

An interesting consequence is obtained when we replace the coefficients in $f = \sum_{n=0}^\infty a_n t^n$ from (30), namely

$$\Big(1 + \frac{t}{2}\Big) f_\alpha\big(\frac{t}{1+t/2}\big) + \frac{1}{\alpha} = 2 f_\alpha\big(\frac{t}{2}\big) + \frac{1+\alpha}{2t^\alpha}\int_0^t z^\alpha f_\alpha\big(\frac{z}{1+z/2}\big)dz, \quad \alpha > -1.$$

By differentiating the identity $\bar{f}_\alpha(x) = \sum_{n=1}^\infty a_n(\alpha)t^n$, where $\bar{f}_\alpha := f_\alpha - \frac{1}{\alpha}$, with respect to α at $\alpha = 0$, we obtain another interesting result. Note that $\bar{f}_0 = l(x)$ and the above transformation leads to (5): $\{a_n(0)\}_1^\infty = \Big\{\frac{1}{1}, 0, \frac{1}{3}\Big(\frac{1}{2}\Big)^2, 0, \frac{1}{5}\Big(\frac{1}{2}\Big)^4, 0, \dots\Big\}$. Also, by (30) it follows $(n+1)a'_{n+1}(0) + a_{n+1}(0) = -na'_n(0)/2$ and one easily represents $\{na'_n(0)2^n\}$ as certain sums. With $z = t/2 = \frac{x}{2-x}$, this gives

$$\begin{aligned} Li_2(x) &= \frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \frac{x^4}{4^2} + \cdots \\ &= 2\Big[\frac{1}{1}\Big(\frac{z^1}{1} - \frac{z^2}{2}\Big) + \Big(\frac{1}{1} + \frac{1}{3}\Big)\Big(\frac{z^3}{3} - \frac{z^4}{4}\Big) + \Big(\frac{1}{1} + \frac{1}{3} + \frac{1}{5}\Big)\Big(\frac{z^5}{5} - \frac{z^6}{6}\Big) + \cdots\Big], \end{aligned}$$

and using that $\arctan^2(z) = \big(\frac{1}{1}\big)z^2 - \frac{1}{2}\big(\frac{1}{1} + \frac{1}{3}\big)z^4 + \frac{1}{3}\big(\frac{1}{1} + \frac{1}{3} + \frac{1}{5}\big)z^6 - + \cdots$ we arrive at

$$Li_2(x) = 2\Big[\frac{1}{1}\frac{z^1}{1} + \Big(\frac{1}{1} + \frac{1}{3}\Big)\frac{z^3}{3} + \Big(\frac{1}{1} + \frac{1}{3} + \frac{1}{5}\Big)\frac{z^5}{5} + \cdots\Big] - L^2(z). \tag{31}$$

The explicit formula $f_\alpha(x) = \sum_{n=0}^\infty \frac{n!}{(\alpha)_{n+1}} \frac{(-x)^n}{(1-x)^{n+1}} =: S_1(x)$, which follows from the Euler transform and $(\alpha)_k := \alpha(\alpha+1)\dots(\alpha+k-1)$, is also of certain interest. This identity gives an acceleration for f if $x < 0$ and especially when $x \approx -1$. Note that the larger is α, the smaller are the terms in the series. Therefore, it makes sense to shift the parameter to the right according to the formula $f_\alpha(x) := \sum_{i=0}^{k-1} \frac{x^i}{i+\alpha} + x^k f_{\alpha+k}(x)$.

Similarly to $l(x)$, $f_\alpha(x)$ has an analytic continuation in $\mathcal{D}_f = \mathbb{C} \setminus [1, \infty)$. From now on $f_\alpha(z)$ will mean this continuation of the series $f_\alpha(z)$. Then, let us justify the domains where the above identities take place. Note that some series representation $S(z)$ of $f_\alpha(z)$ coincide with the function in this connected component of the intersection of the definition domains, which contains $z = 0$. Since the domain of convergence of $S_1(z)$ is $\left|\frac{-z}{1-z}\right| < 1$ and a part of the boundary, depending on α, then $S_1(z)$ represents $f_\alpha(z)$ in the half-plane $Re(z) < 1/2$. Similarly, the domain of convergence of $S_0(\frac{z}{1-z/2})$ is included in $\mathcal{D}_f$, then this series can be used for calculation of $f_\alpha(z)$ in the half-plane $Re(z) < 1$. The remaining part of $\mathcal{D}_f$ can be covered by the following two formulas which are consequences from the relation

$$\alpha f_\alpha(z) = F(1, \alpha; 1 + \alpha; z). \tag{32}$$

For the properties of the hypergeometric function $F(a, b; c; z)$ see [2] and the multiple labels below refer to this book. Now, applying the identity 2.1(17) (which is 2.9(34)) we obtain

$$f_\alpha(z) = \frac{1}{z} f_{1-\alpha}(z^{-1}) + \frac{\pi(-z)^{-\alpha}}{\sin \pi\alpha},$$

where $y^\beta := e^{\beta \log_0(y)}$. According to the above note this relation holds for $z \in \mathbb{C} \setminus [0, \infty)$ and $\alpha \notin \mathbb{Z}$. (When α is an integer, then $f_\alpha(z)$ reduces to $l(z)$ and its analytic continuation is clear.) Another easy consequence of this formula is that when the variable z crosses the segment $(1, +\infty)$ at z_0 in positive direction, then the value of $f_\alpha(z)$ jumps by $2\pi i z_0^{-\alpha}$.

The next transformation changes the argument to $1 - z$ and is very useful for $z \approx 1$. Notice however that f_α belongs to the set of the so-called degenerate cases of the hypergeometric function and many known identities can not be used directly but after a limit passage. Thus, from 2.9(33), applied for $F(1, \alpha; 1 + \alpha + \varepsilon; z)$ with $\varepsilon \to 0$, or directly by 2.3(2) with $l = 0$, we get

$$\begin{aligned} f_\alpha(z) = \sum_{n=1}^{\infty} \Big(\frac{1}{1} - \frac{1}{\alpha} + \frac{1}{2} - \frac{1}{\alpha+1} + \cdots + \frac{1}{n} - \frac{1}{\alpha+n-1}\Big) \frac{(\alpha)_n}{n!}(1-z)^n \\ -\big(\psi(\alpha) + C + \log(1-z)\big) \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!}(1-z)^n, \end{aligned} \tag{33}$$

where $\psi(\alpha)$ is the digamma function and C is the Euler-Mascheroni constant. The relation (33) holds in the domain $\{|z-1| < 1\} \setminus [1, 2)$.

Some other consequences of (32) are:

$$f_\alpha(x) = \int_0^1 \frac{t^{\alpha-1}}{1-xt} dt, \quad Re(\alpha) > 0,\ x \notin [1, \infty),$$

which follows from the Euler integral 2.1.3;

$$f_\alpha(x) = \frac{1}{\alpha -}\ \frac{\alpha^2 x}{\alpha+1 -}\ \frac{1^2 x}{\alpha+2 -}\ \frac{(\alpha+1)^2 x}{\alpha+3\ \ -}\ \frac{2^2 x}{\alpha+4 -}\ \frac{(\alpha+2)^2 x}{\alpha+5\ \ -}\ \cdots\ ,$$

see 2.5.4; next, the forth equality at the definition of u_1 in 2.9 gives

$$f_\alpha(x) = (1-x)^{-\alpha}\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}\frac{y^n}{\alpha+n}, \quad y = \frac{-x}{1-x};$$

and again there, from the second equality, by a limit pass with respect to any parameter of $F(a,b;c;x)$, it follows

$$l(x)f_\alpha(x) = \sum_{n=1}^{\infty}\left(h_n + \frac{1}{\alpha} + \frac{1}{\alpha+1} + \cdots + \frac{1}{\alpha+n-1}\right)\frac{x^n}{\alpha+n}.$$

Another interesting identity is obtained from the relation $f_\nu(z) = \Phi(z,1,\nu)$. Namely, the formula 1.11(9) (which holds for $m=1$ as well), in view of 1.10(11), gives

$$f_\nu(z) = z^{-\nu}\left\{-\sum_{n=1}^{\infty}\frac{B_n(\nu)}{n}\cdot\frac{(\log z)^n}{n!} + \left[\psi(1) - \psi(\nu) - \log\log\frac{1}{z}\right]\right\}, \tag{34}$$

where $B_n(\nu)$ are the Bernoulli polynomials and $|\log z| < 2\pi$.

Finally, we consider the digamma function, because it is closely connected with $f_\alpha(z)$. Indeed, if in place of the divergent series $f_\alpha(1) = \sum_{n=0}^{\infty}\frac{1}{\alpha+n}$ we take $\bar\psi(\alpha) := \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{\alpha+n}\right)$, then $\bar\psi(\alpha) = \psi(\alpha) + C + \frac{1}{\alpha}$. As effective methods for calculation of $\psi(\alpha)$ (and $\bar\psi(\alpha)$) one can use (33) or (34). Also, the formula 1.7(30):

$$\psi(a+z) = \psi(a) + \frac{z}{a} - \frac{1}{2}\frac{z(z-1)}{a(a+1)} + \frac{1}{3}\frac{z(z-1)(z-2)}{a(a+1)(a+2)} - + \cdots$$

can serve for this purpose. Namely, assume that $x = O(1)$ and the value $\psi(x)$ is needed with accuracy 27^{-k}. Then, with $a = k$ and $z = x+k$ take $2k$ summands of the formula. The terms at that place are approximately $(k!)^3/(3k)!$ and decay as $const/3^n$. So, eventually taking several additional summands we stop when the last one becomes less than the required accuracy. Also, the shift formulas 1.7(9): $\psi(k) = h_{k-1} - C$ and 1.7(10): $\psi(x+2k) = \psi(x) + \sum_{j=0}^{2k-1}\frac{1}{x+j}$ are needed for the calculation, and they require $3k$ additional divisions.

We refer to [6] for more recent methods for computation of $\psi(z)$ (and $\Gamma(z)$).

Actually, the series $\psi(\alpha)$ easily can be transformed into a series that converges like $1/n!$, but the problem is that there appear infinitely many unknown constants. For example, such a rearrangement is given by the following formula of type (3)

$$\bar\psi(\alpha) = \frac{c_1\alpha}{1+\alpha} + \frac{c_2\alpha^2}{(1+\alpha)(2+\alpha)} + \frac{c_3\alpha^3}{(1+\alpha)(2+\alpha)(3+\alpha)} + \cdots\ ,$$

where $c_1 = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, $c_2 = 2\sum_{n=2}^{\infty} \frac{n-1}{n^3} = 0.8857...$, $c_3 = 3\sum_{n=3}^{\infty} \frac{(n-1)(n-2)}{n^4} = 0.6102...$, $c_4 = 4\sum_{n=4}^{\infty} \frac{(n-1)(n-2)(n-3)}{n^5} = 0.4663...$, etc. The above series is a consequence of the more general relation

$$\frac{\alpha}{x_1(\alpha+x_1)} + \frac{\alpha}{x_2(\alpha+x_2)} + \frac{\alpha}{x_3(\alpha+x_3)} + \cdots = \frac{c_1\alpha}{\alpha+x_1} + \frac{c_2\alpha^2}{(\alpha+x_1)(\alpha+x_2)} + \cdots,$$

with $\frac{c_1}{x_1} = \sum_{n=1}^{\infty} \frac{1}{x_n^2}$, $\frac{c_2}{x_2} = \sum_{n=2}^{\infty} \frac{x_n-x_1}{x_n^3}$, $\frac{c_3}{x_3} = \sum_{n=3}^{\infty} \frac{(x_n-x_1)(x_n-x_2)}{x_n^4}$, ..., provided the series are convergent. If we set $x_{k+1} = x_{k+2} = \cdots = \infty$, then the relation becomes a polynomial identity, which is not difficult to verify.

Other interesting series are obtained by expanding $\bar{\psi}(\alpha)$ on rational terms containing $\alpha^{(n)} := \alpha(\alpha-1)...(\alpha-n+1)$, for example

$$\begin{aligned}
\bar{\psi}(\alpha) &= 2\left\{\frac{1}{1}\cdot\frac{\alpha}{\alpha+1} + \frac{1}{2}\cdot\frac{\alpha(\alpha-1)}{(\alpha+1)(\alpha+2)} + \frac{1}{3}\cdot\frac{\alpha(\alpha-1)(\alpha-2)}{(\alpha+1)(\alpha+2)(\alpha+3)} + \cdots\right\} \\
&= \sum_{k=1}^{\infty} \frac{\alpha(\alpha^2-1^2)...(\alpha^2-(k-1)^2)}{(1+\alpha)_{2k}} \cdot \frac{(8k-3)\alpha + k(10k-3)}{(2k-1)(2k)} \\
&= \alpha\Big(\frac{1}{\alpha+1} + \frac{1}{2.1}\Big) - \frac{\alpha(\alpha-1)}{2.3}\Big(\frac{1}{\alpha+2} + \frac{1}{2.2}\Big) + \frac{\alpha^{(3)}}{(3)_3}\Big(\frac{1}{\alpha+3} + \frac{1}{2.3}\Big) \\
&\quad - \frac{\alpha^{(4)}}{(4)_4}\Big(\frac{1}{\alpha+4} + \frac{1}{2.4}\Big) + - \ \cdots .
\end{aligned}$$

Note that the last two series converge like a geometrical series with ratio $\frac{1}{4}$.

We shall prove in details only the first identity. We start by proving the formula

$$\frac{\alpha}{\alpha+k} = \sum_{j=1}^{k} c_j(k)\frac{\alpha^{(j)}}{(\alpha+1)_j}, \text{ where } c_j(k) = 2j\frac{(k-1)^{(j-1)}}{(k+1)_j}, \ k = 1,2,3,\ldots \tag{35}$$

To prove the existence of such a representation with certain coefficients we remove the denominators and divide by α arriving to an equality between polynomials of degree $k-1$. Now, choosing the coefficients $\{c_j\}_1^k$ successively by substituting $\alpha = 1,\ldots,k$, the equality follows by the uniqueness of the interpolating polynomial. In order to verify the formula for the coefficients we multiply the identity by $(\alpha+1)_j$ and obtain

$$\frac{\alpha}{\alpha+k}(\alpha+1)_j = \alpha P_{j-1}(\alpha) + (\alpha+1)_j \sum_{i=j+1}^{k} c_i(k)\frac{\alpha^{(i)}}{(\alpha+1)_i},$$

where $P_{j-1}(\alpha)$ is a polynomial of degree $j-1$. Rewriting the last equality as

$$\frac{(1+\alpha)_j - (1-k)_j}{\alpha+k} + \frac{(1-k)_j}{\alpha+k} = P_{j-1}(\alpha) + (\alpha+1)_j \sum_{i=j+1}^{k} c_i(k)\frac{(\alpha-1)^{(i-1)}}{(\alpha+1)_i}$$

and noticing that the second term on the right-hand side vanishes for $\alpha = 1, \dots, j$, we conclude that $P_{j-1}(\alpha) - \frac{(1+\alpha)_j - (1-k)_j}{\alpha+k}$ is the interpolating polynomial for $\frac{(1-k)_j}{\alpha+k}$. In particular the leading coefficient equals

$$c_1(k) + \cdots + c_j(k) - 1 = \frac{(1-k)_j}{\alpha+k}\big[1, 2, \dots, j\big] = \frac{(k-1)^{(j)}}{(k+1)_j},$$

which easily implies the formula for $c_j(k)$.

Now, we substitute $\frac{\alpha}{\alpha+k}$ from (35) into $\bar{\psi}(\alpha) = \sum_{k=1}^{\infty} \frac{\alpha}{k(\alpha+k)}$ and rearrange the summation with respect to the basis $\left\{\frac{\alpha^{(j)}}{(\alpha+1)_j}\right\}$, which is admissible since the double sum has positive terms. Then for the coefficients we get

$$\sum_{k=j}^{\infty} \frac{c_j(k)}{k} = 2j \sum_{k=j}^{\infty} \frac{(k-1)^{(j-1)}}{(k)_{j+1}} = \frac{2j!}{(j)_{j+1}} F(j, j; 2j-1; 1) = \frac{2}{j},$$

where we used that $F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$. The first formula for $\bar{\psi}(\alpha)$ is proved.

For the proof of the second relation one can use the identity

$$\frac{\alpha}{\alpha+n} = \sum_{j=1}^{\lceil n/2 \rceil} \frac{(\alpha+j-1)^{(2j-1)}}{(\alpha+1)_{2j}} \big(e_j(n)\alpha + d_j(n)\big), \quad n = 1, 2, 3, \dots,$$

where $e_j(1)\alpha + d_j(1) = \alpha + 1$ and

$$e_j(n)\alpha + d_j(n) = \frac{n^{(2j-1)}}{(n-j)_{2j+1}} \big((4j-1)(n\alpha - j^2) - j(5j-2)(\alpha - n)\big), \quad n = 2, 3, \dots,$$

while the third one is a consequence of

$$\frac{\alpha}{\alpha+n} = \sum_{j=1}^{n-1} \frac{(-1)^{j-1}}{(n+1)_j} \cdot \alpha^{(j)} + \frac{(-1)^{n-1}}{(n+1)_{n-1}} \cdot \frac{\alpha^{(n)}}{\alpha+n}.$$

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