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NO ZEROS OF THE PARTIAL THETA FUNCTION  
IN THE UNIT DISK

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We prove that for  $q \in (-1, 0) \cup (0, 1)$ , the partial theta function  $\theta(q, x) := \sum_{j=0}^{\infty} q^{j(j+1)/2} x^j$  has no zeros in the closed unit disk.

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1. INTRODUCTION

We consider the partial theta function  $\theta(q, x) := \sum_{j=0}^{\infty} q^{j(j+1)/2} x^j$ , where  $q \in [-1, 1]$  is a parameter and  $x \in \mathbb{R}$  is a variable. In particular,  $\theta(0, x) \equiv 1$ ,  $\theta(1, x) = \sum_{j=0}^{\infty} x^j = 1/(1-x)$  and

$$\theta(-1, x) = \sum_{j=0}^{\infty} (-1)^j x^{2j} + \sum_{j=0}^{\infty} (-1)^{j+1} x^{2j+1} = (1-x)/(1+x^2).$$

For each  $q \in (-1, 0) \cup (0, 1)$  fixed,  $\theta$  is an entire function of  $x$  of order 0.

The name “partial theta function” is connected with the fact that the *Jacobi theta function* equals  $\Theta(q, x) := \sum_{j=-\infty}^{\infty} q^{j^2} x^j$  while  $\theta(q^2, x/q) = \sum_{j=0}^{\infty} q^{j^2} x^j$ . “Partial” refers to the fact that summation in  $\theta$  is performed only from 0 to  $\infty$ . One can observe that

$$\Theta^*(q, x) := \Theta(\sqrt{q}, \sqrt{qx}) = \sum_{j=-\infty}^{\infty} q^{j(j+1)/2} x^j = \theta(q, x) + \theta(q, 1/x)/x.$$

The function  $\theta$  satisfies the relation

$$\theta(q, x) = 1 + qx\theta(q, qx). \quad (1.1)$$

Applications of  $\theta$  to questions concerning asymptotics and modularity of partial and false theta functions and their relationship to representation theory and conformal field theory (see [6] and [4]) explain part of the most recent interest in it. Previously, this function has been studied with regard to Ramanujan-type  $q$ -series (see [25]), statistical physics and combinatorics (see [24]), the theory of (mock) modular forms (see [5]) and asymptotic analysis (see [3]); see also [1].

Another domain in which  $\theta$  plays an important role is the study of section-hyperbolic polynomials. These are real polynomials with all roots real negative and all whose finite sections (i.e., truncations) have also this property, see [9, 21, 22]; the cited papers are motivated by results of Hardy, Petrovitch and Hutchinson (see [7, 8, 23]). Various analytic properties of the partial theta function are proved in [11–20] and other papers of the author.

The basic result of the present text is the following theorem (proved in Section 3):

**Theorem 1.** *For each  $q \in (-1, 0) \cup (0, 1)$  fixed, the function  $\theta$  has no zeros in the closed unit disk  $\overline{\mathbb{D}}_1$ .*

In the next section we discuss the question to what extent Theorem 1 proposes an optimal result. In Section 4 we make comments and formulate some open questions.

## 2. OPTIMALITY OF THE RESULT

### 2.1. THE THEOREM OF ENESTRÖM-KAKEYA

For  $q \in (0, 1)$ , the theorem of Eneström-Kakeya about polynomials with positive coefficients (see [2]) implies that the modulus of each root of a polynomial  $a_0 + a_1x + \dots + a_nx^n$ ,  $a_j > 0$ , is not less than  $\min_j |a_{j-1}/a_j|$ . When this polynomial equals  $1 + qx + \dots + q^{n(n-1)/2}x^{n-1}$ , the minimum equals  $1/q$ . Thus all zeros of all finite truncations of  $\theta(q, \cdot)$  (and hence all zeros of  $\theta$  itself) lie outside the open disk  $\mathbb{D}_{1/q}$ .

Hence for  $q \in (0, 1)$  (but not for  $q \in (-1, 0)$ ), Theorem 1 follows from the theorem of Eneström-Kakeya. A hint how to obtain for  $q \in (-1, 0)$  a disk of a radius tending to  $\infty$  as  $q \rightarrow 0^-$  and free from zeros of  $\theta$  is given in Remark 4.

**Remark 2.** For  $q \in (-1, 0)$ , it is not true that  $\theta(q, \cdot)$  has no zeros inside the disk  $\mathbb{D}_{1/|q|}$ . Indeed, the function  $\theta(-0.4, \cdot)$  has a zero  $1.96\dots < 1/0.4 = 2.5$ . More generally, the zero of  $\theta(q, \cdot)$  closest to the origin can be expanded in a Laurent series (convergent for  $0 < |q|$  sufficiently small) of the form  $-1/q - 1 + O(q)$ , see [11]. For  $q \in (-1, 0)$  and  $|q|$  sufficiently small, this number belongs to the interval  $(0, 1/|q|)$ . See also [20], where the zero set of  $\theta$  is illustrated by pictures.

2.2. OPTIMALITY WITH RESPECT TO THE PARAMETER  $q$

(1) This result cannot be generalized in the case when  $q$  and  $x$  are complex. Indeed, suppose that  $q \in \mathbb{D}_1$  and  $x \in \mathbb{C}$ . Then the function  $\theta$  has no zeros  $x$  with  $|x| < 1/2$ . In fact, it has no zero for  $|x| \leq 1/2|q|$ , see [10, Proposition 7]. On the other hand, the radius of the disk in the  $x$ -space centered at 0 in which  $\theta$  has no zeros for any  $q \in \mathbb{D}_1$  is not larger than  $0.56\dots$ . Indeed, consider the series  $\theta$  with  $q = \omega := e^{3\pi i/4}$ . It equals

$$\left( \sum_{j=0}^7 \omega^{j(j+1)/2} x^j \right) / (1 - x^8).$$

Its numerator has a simple zero  $x_* := 0.33\dots + 0.44\dots i$  whose modulus equals  $0.56\dots$ . Hence for  $\rho \in (0, 1)$  sufficiently close to 1, the function  $\theta(\rho e^{3i\pi/4}, \cdot)$  has a zero close to  $x_*$ . To see this one can fix a closed disk  $\mathcal{D}$  about  $x_*$  of radius  $< 0.1$  in which  $x_*$  is the only zero of  $\theta(e^{3i\pi/4}, \cdot)$ . As  $\rho$  tends to  $1^-$ , the modulus of the difference  $\theta(e^{3i\pi/4}, x) - \theta(\rho e^{3i\pi/4}, x)$  tends uniformly to 0 for  $x \in \partial\mathcal{D}$  (the border of  $\mathcal{D}$ ), because the series  $\theta$  converges uniformly for  $|x| < |x_*| + 0.1$ ,  $|q| \leq 1$ . The Rouché theorem implies that the function  $\theta(\rho e^{3i\pi/4}, \cdot)$  has the same number of zeros in  $\mathcal{D}$  (counted with multiplicity) as the function  $\theta(e^{3i\pi/4}, \cdot)$ .

(2) Set  $q := |q|e^{i\phi}$ . We show that there exists no interval (i.e., arc)  $J$  on the unit circle centered at 1 or  $-1$  and such that for  $\phi \in J$  and  $|q| < 1$ , the zeros of  $\theta(q, \cdot)$  are all of modulus  $\geq 1$ . Suppose that  $n \in \mathbb{N}$ ,  $n > 2$ , and that  $\omega$  is a primitive root of unity of order  $n$ . If  $n$  is odd, then the sequence of numbers  $\omega^{k(k+1)/2}$  is  $n$ -periodic, because  $(n + 1)/2 \in \mathbb{N}$ , and one obtains

$$\theta(\omega, x) = P(x)/(1 - x^n), \quad P := \sum_{j=0}^{n-1} a_j x^j, \quad a_j = \omega^{j(j+1)/2}.$$

If  $n$  is even, then this sequence is clearly  $(2n)$ -periodic, but it is not  $n$ -periodic, because  $\omega^{n(n+1)/2} = -1$ . One has

$$\theta(\omega, x) = Q(x)/(1 - x^{2n}), \quad Q := \sum_{j=0}^{2n-1} b_j x^j, \quad b_j = \omega^{j(j+1)/2}.$$

The polynomials  $P$  and  $Q$  are self-reciprocal, i.e.,  $a_{(n-1)/2-s} = a_{(n-1)/2+s}$  and  $b_{(2n-1)/2-s} = b_{(2n-1)/2+s}$ . Indeed, for the polynomial  $P$  this follows from

$$\begin{aligned} ((n-1)/2 - s)((n-1)/2 - s + 1)/2 &\equiv (n - (n-1)/2 + s)(n - (n-1)/2 + s - 1)/2 \\ &= ((n-1)/2 + s)((n-1)/2 + s + 1)/2 \pmod{[n]}. \end{aligned}$$

For the polynomial  $Q$  one gets

$$\begin{aligned} ((2n-1)/2 - s)((2n-1)/2 - s + 1)/2 &\equiv (2n - (2n-1)/2 + s)(2n - (2n-1)/2 + s - 1)/2 \\ &= ((2n-1)/2 + s)((2n-1)/2 + s + 1)/2 \pmod{[2n]}. \end{aligned}$$

We show that at least one root of the polynomial  $P$  and at least one root of  $Q$  belong to the interior of the unit disk. Indeed, these polynomials are monic and  $P(0) = Q(0) = 1$ . The product of their roots being equal to  $\pm 1$ , the only possibility for  $P$  and  $Q$  not to have roots in  $\mathbb{D}_1$  is all their roots to be of modulus 1. These polynomials are self-reciprocal, so  $P(z) = 0$  (resp.  $Q(z) = 0$ ) implies  $P(1/z) = 0$  (resp.  $Q(1/z) = 0$ ). But if  $|z| = 1$ , then  $1/z = \bar{z}$ . This means that  $P$  and  $Q$  can have as roots either  $\pm 1$  or complex conjugate pairs, i.e.,  $P$  and  $Q$  must be real which is false as their coefficients of  $x$  equal  $\omega \neq \pm 1$ .

So  $P$  and  $Q$  have each at least one root in  $\mathbb{D}_1$ . As in part (1) of this subsection one deduces that for  $|q|$  sufficiently close to 1 and for  $e^{i\phi} = \omega$ , the function  $\theta(q, \cdot)$  has a zero in  $\mathbb{D}_1$ . Primitive roots are everywhere dense on the unit circle. This implies the absence of an interval  $J$  as above.

### 2.3. OPTIMALITY WITH RESPECT TO THE VARIABLE $x$

Suppose first that  $q \in (-1, 0)$ . Then in the formulation of Theorem 1 one cannot replace the unit disk by a disk of larger radius. Indeed, the zero of the numerator of  $\theta(-1, x)$  (which equals 1) is the limit as  $q$  tends to  $-1^+$  of the smallest positive zero of  $\theta(q, x)$ , see [20, Part (2) of Theorem 3], so in any disk  $\mathbb{D}_{1+\varepsilon}$ ,  $\varepsilon > 0$ , there is a zero of  $\theta$  for some  $q \in (-1, 0)$ .

Suppose now that  $q \in (0, 1)$ .

**Conjecture 3.** *Theorem 1 does not hold true if one replaces in its formulation the unit disk by a disk of larger radius.*

The following numerical example shows why this conjecture should be considered plausible. Set  $\theta_{100} := \sum_{j=0}^{100} q^{j(j+1)/2} x^j$  (the 100th truncation of  $\theta$ ). For  $q = 0.98$ , the function  $\theta_{100}(0.98, \cdot)$  has a zero  $\lambda_0 := 1.209 \dots + 0.511 \dots i$ , of modulus  $1.312 \dots$ . For  $q = 0.98$  and  $x = 1.32$ , the first two terms of  $\theta$  which are not in  $\theta_{100}$  equal  $y_{101} := 7.407 \dots \times 10^{-33}$  and  $y_{102} := 1.270 \dots \times 10^{-33}$ , respectively. Their ratio is  $y_{101}/y_{102} > 5.5$  and the moduli of the terms of  $\theta$  decrease faster than a geometric progression. Hence for  $|x| < 1.32$ , one has

$$T_0 := |\theta(0.98, x) - \theta_{100}(0.98, x)| < y_{101}/(1 - 5.5^{-1}) = 9.053 \dots \times 10^{-33}.$$

On the other hand,  $\Lambda_0 := (\partial\theta/\partial x)(0.98, \lambda_0) = 27.180 \dots + 18.959 \dots i$  with  $|\Lambda_0| > 33$ . Thus one should expect to find a zero of  $\theta(0.98, \cdot)$  close to  $\lambda_0$  (the truncated terms are expected to change the position of  $\lambda_0$  by  $\approx T_0/|\Lambda_0|$  which quantity is of order  $10^{-34}$ ). So in the formulation of Theorem 1 one should not be able to replace the unit disk by a disk of radius larger than 1.32.

### 3. PROOF OF THEOREM 1

We remind first that the *Jacobi triple product* is the identity

$$\Theta(q, x^2) = \prod_{m=1}^{\infty} (1 - q^{2m})(1 + x^2 q^{2m-1})(1 + x^{-2} q^{2m-1})$$

which implies  $\Theta^*(q, x) = \prod_{m=1}^{\infty} (1 - q^m)(1 + xq^m)(1 + q^{m-1}/x)$ . Thus

$$\prod_{m=1}^{\infty} (1 - q^m)(1 + xq^m)(1 + q^{m-1}/x) = \theta(q, x) + \theta(q, 1/x)/x. \tag{3.1}$$

Suppose that  $q \in (-1, 0) \cup (0, 1)$ ,  $x_0 \in \mathbb{C}$ ,  $|x_0| = 1$  (hence  $\overline{x_0} = 1/x_0$ ), and that  $\theta(q, x_0) = 0$ . The coefficients of  $\theta$  being real, one has  $\theta(q, \overline{x_0}) = \overline{\theta(q, x_0)} = 0$ , so the right-hand side of equation (3.1) equals 0 for  $x = x_0$ . However for  $x = x_0$ , the left-hand side vanishes only for  $x_0 = -1$ .

For  $q \in (0, 1)$ , one has  $\theta(q, -1) = \sum_{j=0}^{\infty} (-1)^j q^{j(j+1)/2}$ , and the latter function takes only values from the interval  $(1/2, 1)$ , with  $\lim_{q \rightarrow 1^-} = 1/2$ , see [10, Propositions 14 and 16]. For  $q \in (-1, 0)$ , one sets  $u := -q$ , so

$$\begin{aligned} \theta(q, -1) &= \theta(-u, -1) = 1 + u - u^3 - u^6 + u^{10} + u^{15} - u^{21} - u^{28} + \dots \\ &= 1 - u^3 + u^{10} - u^{21} + \dots + u - u^6 + u^{15} - u^{28} + \dots > 0, \end{aligned}$$

because this is the sum of two Leibniz series with positive initial terms. Thus for  $q \in (-1, 0) \cup (0, 1)$ , the partial theta function has no zeros of modulus 1.

For  $-1/2 < q < 1/2$ , one has  $\theta(q, x) \neq 0$  for any  $x \in \overline{\mathbb{D}_1}$ , because

$$|\theta(q, x)| \geq 1 - |q| - |q|^3 - |q|^6 - \dots \geq 1 - |q| - |q|^2 - |q|^3 - \dots = (1 - 2|q|)/(1 - |q|) > 0.$$

As the parameter  $q$  varies in  $(0, 1)$  or in  $(-1, 0)$ , the zeros of  $\theta$  depend continuously on  $q$ . For  $|q| < 1/2$ , there are no zeros of  $\theta$  in  $\overline{\mathbb{D}_1}$  and for  $q \in (-1, 0) \cup (0, 1)$ , no zero of  $\theta$  crosses  $\partial\mathbb{D}_1$  (the border of the unit disk). Hence for  $q \in (-1, 0) \cup (0, 1)$ , there are no zeros of  $\theta$  in  $\overline{\mathbb{D}_1}$ .

**Remark 4.** One can prove that for  $|q| \leq 0.4$ , the function  $\theta(q, \cdot)$  has no zeros in the closed disk  $\overline{\mathbb{D}_{1/\sqrt{|q|}}}$ . Indeed,

$$|\theta(q, 1/\sqrt{|q|})| \geq 1 - \sum_{j=1}^{\infty} |q|^{j(j+1)/2 - j/2} = 1 - \sum_{j=1}^{\infty} |q|^{j^2/2} \geq 1 - \sum_{j=1}^{\infty} 0.4^{j^2/2} = 0.19\dots > 0.$$

#### 4. COMMENTS AND OPEN QUESTIONS

##### 4.1. THE CASE $q \in (0, 1)$

In Figure 1 we show the images for  $q = 0.2$  (the smaller oval) and for  $q = 0.7$  (the larger oval) of the unit circle in the  $x$ -plane under the mapping  $x \mapsto \theta(q, x)$ , together with the vertical line  $\operatorname{Re} x = 1/2$ .

It would be interesting to know whether:

1. for any  $q \in (0, 1)$ , the image of the unit circle is a convex oval about the point  $(1, 0)$  and belonging to the half-plane  $\operatorname{Re} x > 1/2$ ;

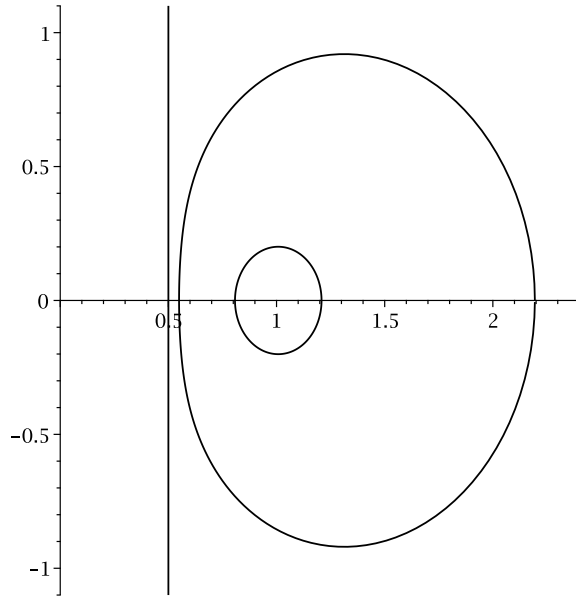


Figure 1. The vertical line  $\operatorname{Re} x = 1/2$  and the images of the unit circle under the map  $x \mapsto \theta(q, x)$  for  $q = 0.2$  and  $q = 0.7$

2. for  $0 < q_1 < q_2 < 1$ , the image of the unit circle for  $q = q_1$  lies inside its image for  $q = q_2$ .

These questions are motivated by the fact that for  $q = 1$ , one has  $\theta(1, x) = 1/(1-x)$ , and for  $|x| = 1$ , it is true that  $\operatorname{Re}(1/(1-x)) = 1/2$ , i.e., the vertical line  $\operatorname{Re}(1/(1-x)) = 1/2$  is the image of the unit circle for  $q = 1$ ; on the other hand, the point  $(1, 0)$  is the image of the unit circle for  $q = 0$ .

#### 4.2. THE CASE $q \in (-1, 0)$

In Figure 2 we show the images for  $q = -0.2$  (small oval in dashed line),  $q = -0.53$  (closed contour in dotted line),  $q = -0.7$  (curve with self-intersection in dashed line) and  $q = -0.85$  (curve with self-intersection in solid line) of the unit circle in the  $x$ -plane under the mapping  $x \mapsto \theta(q, x)$ .

The following questions are natural to ask:

3. Is it true, and for which value of  $v \in (0, 1)$ , that for  $q \in (-v, 0)$ , the corresponding image is a convex oval about the point  $(1, 0)$ ?
4. Is it true that for  $q \in (-w, -v)$ ,  $-1 < -w < -v < 0$ , the corresponding image changes convexity twice “at its right” (as this seems to be the case of the curve given in dotted line)?

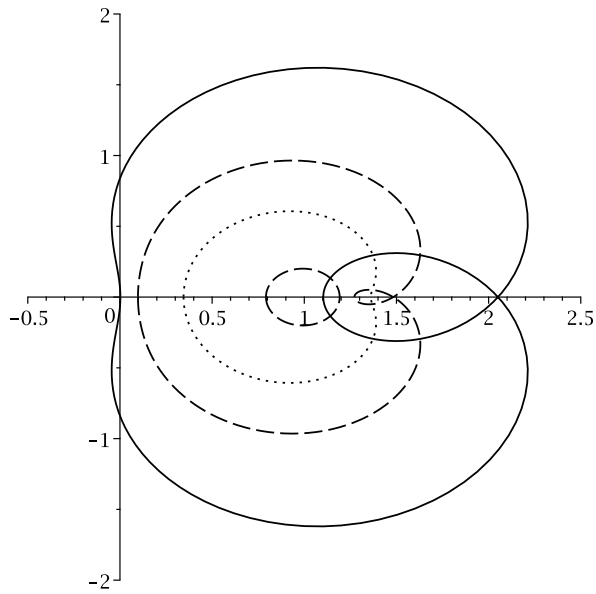


Figure 2. The images of the unit circle under the map  $x \mapsto \theta(q, x)$  for  $q = -0.2$ ,  $q = -0.53$ ,  $q = -0.7$ , and  $q = -0.85$

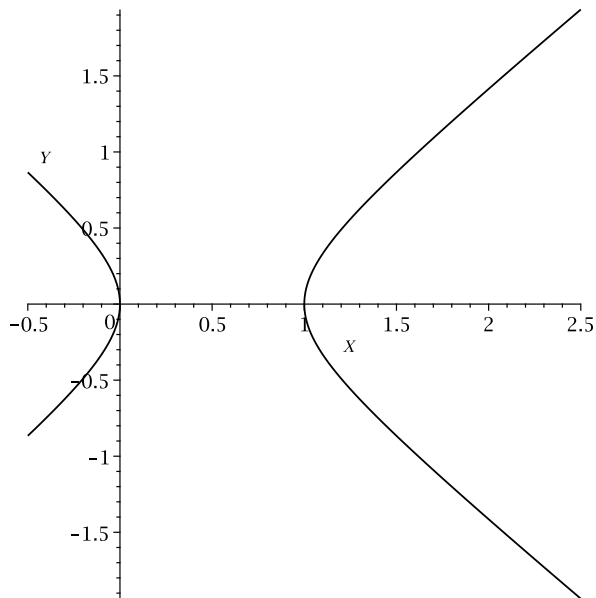


Figure 3. The image of the unit circle under the map  $x \mapsto \theta(q, x)$  for  $q = -1$

5. Is it true that for  $q \in (-1, -w)$ , the image has a self-intersection point? One can expect that for  $q = -w$ , the image has a cusp point.
6. Is it true that for  $q \in (-1, -w')$ ,  $-1 < -w' < -w$ , the image has still self-intersection and changes convexity twice “at its left”?
7. Is it true that for  $q \in (-1, -w'')$ ,  $-1 < -w'' < -w'$ , the image has still self-intersection, changes convexity twice “at its left” and intersects the vertical axis at four points? (For  $q = w''$ , the image is supposed to have two tangencies with the vertical axis.)
8. Is it true that these are all transformations which the image undergoes for  $q \in (-1, 0)$ ?
9. Is it true that for  $-1 < q_2 < q_1 < 0$ , the image of the unit circle for  $q = q_1$  lies inside its image for  $q = q_2$ ? “Inside” means “inside the contour excluding (for  $q_2 > w$ ) the loop”.

It should be observed that for values of  $q$  close to  $-1$ , the image seems to pass through the origin. In reality, it passes very close to it, but nevertheless to its right, according to Theorem 1. The image of the unit circle for  $q = -1$  is the hyperbola  $Y^2 - X^2 - X = 0$ , where  $X := \operatorname{Re} x$  and  $Y := \operatorname{Im} x$ , see Figure 3. (The centre of the hyperbola is at  $(1/2, 0)$ , its asymptotes are the lines  $Y = \pm(X - 1/2)$ .) Following a similar logic one can assume that the point  $(1, 0)$  remains in the exterior of the loop of the image (the loop existing for  $q > w$ ). The proximity of the image to the origin makes it seem unlikely that one could prove the absence of zeros of  $\theta$  in a disk of a radius larger than 1 (for all  $q \in (-1, 0)$ ).

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