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# IN MEMORIAM
# IORDAN IORDANOV (1948–2021)

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Iordan Iordanov passed away unexpectedly on October 3, 2021. His loss is deeply felt by the academic community and friends around the world. The current work provides an overview of his academic career and achievements and surveys his research in mathematics, economics and finance.

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## 1. Introduction: Iordan Iordanov's life and work

Iordan Iordanov suddenly passed away on October 3, 2021. Bulgarian academia lost a dedicated educator, passionate researcher, erudite scholar and dear friend. His absence will be especially felt in the community specializing in mathematical modelling applied to economics and finance, as Iordan was one of the pioneers of the field in post-communist Bulgaria.

Iordan Iordanov was born on February 27, 1948 in Sofia. He grew up in the town of Sapareva Bania and, after graduating from high school in Stanke Dimitrov (present-day Dupnitsa), went on to study mathematics at Sofia University "St. Kliment Ohridski". There he was noticed by Prof. Todor Genchev, who encouraged his interest in the field of partial differential equations. Iordan was actively involved in academic life even before graduating in mathematics in 1971, teaching seminars in his student days. Following his graduation, he assumed a teaching position in the Department of Differential Equations of the Faculty of Mathematics and Informatics at Sofia University.

Iordan defended his PhD dissertation titled "Variational inequalities for operators in partial derivatives" in 1980 and obtained habilitation as associate professor at the Faculty of Mathematics and Informatics of Sofia University "St. Kliment Ohridski" in 1983. At the time his research focused on variational inequalities and analysis of singularities of solutions to certain classes of partial differential equations. More details on his mathematical works are provided in Section 2.

As a teacher of mathematics, Iordan had a reputation of being stern, yet fully devoted to passing on the science and the craft. He taught a number of courses, including on partial differential equations, mathematical analysis for students in Physics and theory of distributions. He was noted for always striving to make accessible even the most complex concepts and seeking to present the basics of the respective subjects in line with modern practices and approaches. One of the authors of the present work, Vladimir Georgiev, recollects the following:

> Iordan had a really well-prepared and attractive course on Partial Differential Equations. Typically, this course was not considered very easy by the students. However, Iordan succeeded to attract several students. Among them was Angel Ivanov, who did a master's thesis with Iordan Iordanov and then won a PhD position at University of Pisa. This was the first Bulgarian PhD student at University of Pisa. The role of Iordan in the preparation of Angel as a PhD student was really crucial.

In the early 1990s Iordan Iordanov embarked on a new journey. He was one of the group of enthusiasts who with a lot of effort and energy re-established the Faculty of Economics and Business Administration at Sofia University in 1990. He was very active both academically and administratively, serving as vice-dean and acting dean of the faculty during various periods.

During the early years of the building of the Faculty of Economics and Business Administration Iordan made key contributions to the creation of the curricula of the two mathematics courses and the quantitative methods course taught at the faculty, in line with state-of-the-art practices in teaching these disciplines. He is the main author of the textbook that is still used in the teaching of mathematics at the Faculty of Economics and Business Administration. Iordan contributed to assembling and expanding the team of lecturers at the faculty and to introducing French-language programmes. He actively supported inter-university cooperation through student and lecturer exchanges.

In the 2000s Iordan redirected his efforts to a new project – the development of a master's programme on mathematical modelling in economics at the Faculty of Mathematics and Informatics. Together with a team of collaborators he researched best practices in the teaching of mathematical modelling applied to economics. He tirelessly worked to put together a curriculum that is appropriate for the audience at the Faculty of Mathematics and Informatics, as well as to ensure there is a team of lecturers providing the teaching. The program was launched successfully with Iordan as its first director. It continues to run to this day.

As a fellow academic, Iordan Iordanov was extremely cooperative and supportive. He was always ready to discuss, provide hints or materials, explain the finer points of an intricate issue and generally spare the time to delve into a problem and push it forward. His colleagues, students and collaborators frequently noted his generosity, meticulousness and drive to attain useful, high-quality results. These qualities carried over to organisational and administrative tasks, where he was hard-working, result-oriented and always caring and respectful to people.

This is what one of the authors, Nikolay Chervenov, shares about working with Iordanov:

> I met Iordanov for the first time in Vasil Tsanov's office, when I was a master's student in Equations of Mathematical Physics. He needed a student to assist him with the seminars for his courses. We connected right away, and with his impressive teaching flair, I soon became his assistant for Calculus, Differential Equations and Partial Differential Equations, Complex analysis. Two years later, I naturally chose him as my doctoral supervisor, as we had already established a good rapport.
>
> Copula functions, which were the core of most of our joint works, caught his interest after he saw Prof. Leda Minkova's open notes. He was drawn to the expression of the copula density as its mixed partial derivative. As a differential equations expert, he quickly wondered about the existence of this derivative and how to build a family of copulas as a solution to differential equations. A result that would be completely new in the field of copulas.

Iordan Iordanov did not confine his efforts to popularise mathematical modelling in economics to Sofia University only. He taught various courses at the Technical University of Sofia and wrote textbooks on macroeconomics and mathematical models in economics ([43, 44]). Iordan also published a number of papers on various economic and finance issues, including on the methodology of teaching economics through a quantitative lens (as detailed in Sections 3 and 4). He consulted for various institutions, including the Bulgarian National Bank, the Agency for Economic Analysis and Forecasting, as well as private entities.

After formally retiring from Sofia University, Iordan never lost momentum. He continued teaching at the Faculty of Mathematics and Informatics and at the Technical University of Sofia. He also assumed a position at the International Business School, where he contributed to the curriculum on mathematics and quantitative methods.

Iordan's interests and erudition extended far beyond mathematics and economic modelling. He spoke several foreign languages, dabbled in psychiatry and health practices, and was surprisingly skilled at gardening and different crafts. He was an avid lover of travel and nature, never missing an opportunity to experience new sights and places.

## 2. Mathematical contributions

Iordan's initial research interests included the regularity of solutions to some elliptic variational problems. The works [6–12] treat different aspects of this topic.

Typically, the problem is associated with a bounded domain $\Omega \subset \mathbb{R}^n$. The elliptic operator of type

$$Lv \equiv -\sum_{i,j=1}^{n} \left(a_{ij}(x) v_{x_i}\right)_{x_j} + c(x)v$$

defines the bilinear form

$$a(u,v) \equiv \int_\Omega \left( \sum_{i,j=1}^{n} a_{ij} u_{x_i} v_{x_j} \right) \mathrm{d}x + \int_\Omega cuv \,\mathrm{d}x. \tag{2.1}$$

A classical result of Lions and Stampacchia [37] treats a generic quadratic form $a$ in a Hilbert space $H$. In the case of one convex set $K$ in $H$ the result in this article gives a unique element $u \in K$, such that

$$a(u, v-u) \geq 0 \quad \text{for any} \quad v \in K.$$

As second important point in his study Iordan chose the comparison approach proposed in the work of Lewy and Stampacchia [36], where the following comparison notion is introduced.

Let $E$ be a closed set of $\overline{\Omega}$. An element $u$ of $H^1(\Omega)$ will be said to be nonnegative on $E$ in the sense of $H^1(\Omega)$ if there exists a sequence $\{u_m\}$ of functions of $C^1(\overline{\Omega})$ such that

(i) $u_m \geq 0$ on $E$,

(ii) $u_m \to u$ in $H^1(\Omega)$ as $m \to \infty$.

More generally, we say that, for $u \in H^1(\Omega)$, $v \in H^1(\Omega)$, $u \geq v$ on $E$ in the sense of $H^1(\Omega)$ if $u - v \geq 0$ on $E$ in the sense of $H^1(\Omega)$.

In the previous definition condition (ii) may be replaced by (ii)′ $u_m \to u$ weakly in $H^1(\Omega)$. In fact, if there exists a sequence $\{u_m\}$ of $C^1(\overline{\Omega})$ functions satisfying (i) and (ii)′, then, by a well-known theorem of Banach-Saks, there exists a sequence $\{u'_m\}$ of positively weighted means of $\{u_m\}$ which satisfies conditions (i) and (ii).

We turn to the quadratic form $a(u, v)$ defined in (2.1) for $u, v \in H^1(\Omega)$. In a series of works [6–12] Iordan studied appropriate generalization of existence and uniqueness of solutions of variational inequalities for the case of two closed subsets $E$ and $F$ in $\Omega$ with nonzero measures.

More precisely, given any $\psi \in H_0^1(\Omega)$ one can define

$$N = \sup_E \psi, \quad M = \inf_F \psi. \tag{2.2}$$

Iordan used the following variant of the Stampacchia comparison definition. One can say that $v, u \in H_0^1(\Omega)$ satisfy the relation $v \geq u$ in sense of $H_0^1(\Omega)$ in some subsets in $\Omega$, if there exist sequences $\{v_n\}$ and $\{u_n\}$ approximating $v$ and $u$ in $H^1(\Omega)$ and for all $n$ satisfying $v_n \geq u_n$ in the chosen subset. Then one defines the following convex subsets in $H_0^1(\Omega)$:

$$K_1 = \left\{ v \mid v \in H_0^1(\Omega),\ v \geq \psi \text{ in } E \text{ in the sense of } H_0^1(\Omega) \right\}, \tag{2.3}$$

$$K_2 = \left\{ v \mid v \in H_0^1(\Omega),\ v \leq \psi \text{ in } F \text{ in the sense of } H_0^1(\Omega) \right\}. \tag{2.4}$$

In this way Iordan faced a situation different from the classical case of one convex set $K$ in a Hilbert space $H$ analysed in [37]. In fact, in his works Iordan generalized this result for the case of two convex sets $K_1$, $K_2$.

The results obtained in the works of Iordan give existence and uniqueness of solution $u \in K_2$ to the following variational inequality

$$a(u, v - u) \geq 0 \quad \text{for any} \quad v \in K_2. \tag{2.5}$$

The main difficulty is the control of the approximation sequence for the variational inequality (2.5) near $E \cap F$ and the weak regularity of the boundary of this set. In a series of lemmas Iordan succeeded to define the approximating sequence and with appropriate extension outside the boundary of $E \cap F$.

The case when the Sobolev space is $H^{1,p}$ type is considered in [12–14, 38]. Concerning the regularity of the solution Iordan studied and applied the method developed by G. Stampacchia in [41].

Iordan obtained the following important result.

**Theorem 2.1.** *Assume the boundary $\partial\Omega$ of the domain $\Omega$ is in the class $C^{1,1}$ and that $\varphi, \psi_1 \in H^{1,p}(\Omega)$ with $p > n$. We assume further that*

$$\varphi < \psi \text{ on } \Gamma = \overline{\Omega \cap \partial (E \cap F)}$$

*and $\Gamma \neq \varnothing$. We assume further that the boundaries $\partial\left(\overline{\Omega} \setminus E\right)$ and $\partial\left(\overline{\Omega} \setminus F\right)$ are Lipschitz continuous and that the coefficients $a_{ij} = a_{ji}$, $i, j = 1, \ldots, n$, are Lipschitz continuous in $\overline{\Omega}$.*

*Then the solution $u$ to the problem (2.5) is in the space*

$$C^{0,\alpha}(\overline{\Omega}), \quad \alpha \in \left(0, 1 - \frac{n}{p}\right].$$

This result and its generalizations can be found in [14].

Furthermore, Iordan Iordanov was interested in the problem of creation of singularities for some non-strictly hyperbolic systems. At this time he was discussing often the propagation of singularities of solutions of PDE's with Vesselin Petkov.

The work [16] deals with the Cauchy problem for the semilinear non-strictly hyperbolic system of type

$$(\partial_t + \lambda_j(x,t)\partial_x)\, u_j = g_j\left(x, t, u_1, \ldots, u_n\right), \quad j = 1, \cdots, n, \tag{2.6}$$

where

$$\lambda_j \in C^\infty\left(\mathbb{R}^2\right), \quad g_j \in C^\infty\left(\mathbb{R}^{n+2}\right).$$

At $t = 0$ the first speed velocity coefficient satisfies

$$\lambda_1 = t^{2q-1}\alpha(x,t), \quad \alpha \in C^\infty\left(\mathbb{R}^2\right), \quad \alpha(x,t) \geq \alpha_1 = \text{const} > 0, \quad q \in \mathbb{N}.$$

The second term $\lambda_2$ is identically 0. For other speed velocities the assumption of non-degeneracy is the following one

$$\inf_{\substack{i,j=1,\ldots,n \\ i<j,\,(i,j)\neq(1,2)}} |\lambda_i(x,t) - \lambda_j(x,t)| \geq \delta = \text{const} > 0.$$

Iordan studied the propagation of anomalous singularities; these can exist along the forward characteristics from the points of singular support of the initial data.

The interaction of the singularities propagating along two characteristics $C_1$, $C_2$ crossing at some point could give rise of new singularities (the so-called anomalous singularities) propagating along the outgoing transversal characteristics starting from that point. The solution is supposed smooth along the corresponding backward transversal characteristics. The anomalous singularities are weaker than the initial ones and in the case of weakly hyperbolic semilinear systems the larger the order of contacts between the characteristics $C_1$, $C_2$ carrying out the initial singularities, the more regular the solutions.

In the works [16, 39] an example of a weakly hyperbolic system is studied. In a sufficiently small neighbourhood of the origin in $\mathbb{R}^2_{(x,t)}$ the following weakly hyperbolic system is considered:

$$\begin{aligned} &Xu \equiv \left(\partial_t + pt^{p-1}\partial_x\right) u = 0, \\ &\partial_t v = u, \\ &Dz \equiv (\partial_t + \partial_x)\, z = f(u,v), \end{aligned}$$

where $p$ is a positive integer and the initial data are given at $t = -T$, $T > 0$,

$$u(x, -T) = u_0(x), \quad v(x, -T) \equiv z(x, -T) \equiv 0.$$

The characteristics $x = t^p + \text{const}$ of $X$ and $x = \text{const}$ of $\partial_t$ have order of contact $p - 1$ at points on the line $t = 0$ and the characteristics of $D(x = t + \text{const})$ are transversal to them. Denoting by $C_i$, $i = 1, 2, 3$, the characteristics $x = t^p$, $x = 0$ and $x = t$, passing through the origin $(0, 0)$ one can define $C_i^+$ as the part of $C_i$ for $t > 0$. It is assumed that the function $u_0(x)$ has a jump discontinuity of order $k$ ($k \geq 0$ integer) at $x_0 = T^p$.

Iordan valued highly working with young researchers. The work of Angel Ivanov at University of Pisa referenced in Section 1 was done practically in collaboration with Iordan. As a result Angel had two important publications [4, 5].

Consider the following Schrödinger equation with potential perturbation:

$$i\partial_t u - \Delta u + Vu = 0, \quad \Delta = \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2, \tag{2.7}$$

$$u(0, x) = u_0(x), \quad x \in \mathbb{R}^3. \tag{2.8}$$

Here $V = V(x)$ is a real-valued potential that satisfies the assumption

$$\|V\|_{L^{(3/2,\infty)}} \leq \delta_0, \tag{2.9}$$

where $L^{(p,q)}$ are standard Lorentz spaces, $L^{(p,\infty)}$ is the weak $L^p$ space. For $\delta_0 > 0$ sufficiently small one can define the bilinear form

$$Q(u, v) = (\nabla u, \nabla v)_{L^2(\mathbb{R}^3)} + \int_{\mathbb{R}^3} V(x) u(x) \overline{v(x)} \, \mathrm{d}x \tag{2.10}$$

on $\dot{H}^1(\mathbb{R}^3)$. The Friedrichs extension of the quadratic form, implies that $-\Delta + V$ has dense domain $D = H^2(\mathbb{R}^3)$ such that, if $f \in D$, then $(-\Delta + V)f \in L^2$ and $Vf \in L^2$. Moreover, $-\Delta + V$ with dense domain $D$ is a self-adjoint operator. The existence and mapping properties of the wave operators

$$\Omega_\pm = s - \lim_{t \to \pm\infty} e^{itH} e^{-itH_0}, \tag{2.11}$$

where $H_0 = -\Delta$, $H = -\Delta + V$, are presented in [33]. The crucial assumption is that $V$ belongs to the Kato class (see, for example, [33]). It is easy to see that potential of type

$$V(x) = \frac{W\left(\frac{x}{|x|}\right)}{|x|^2}, \tag{2.12}$$

where $W \in L^\infty(S^2)$, does not belong to the Kato's class (for example we can take $W(x/|x|) = 1$).

The main goal of this line of work was to prove the existence and mapping properties of the wave operator $\Omega_\pm$ for the larger class of potentials $V$, satisfying only (2.9). This problem is closely connected with another classical problem, namely the Strichartz-type estimates for the corresponding inhomogeneous Cauchy problem

$$i\partial_t u - \Delta u = F, \quad u(0) = f. \tag{2.13}$$

We shall call the pair $\left(\frac{1}{p}, \frac{1}{q}\right)$ sharp admissible (see [35] for this notion and the properties of sharp admissible pairs), if it satisfies the condition

$$\frac{3}{4} = \frac{1}{p} + \frac{3}{2q}, \quad 2 \leq p \leq \infty. \tag{2.14}$$

The main result is given in the next theorem.

**Theorem 2.2.** *If $(p, q)$ and $(\tilde{p}, \tilde{q})$ satisfy (2.14), then the solution to the Cauchy problem*

$$\begin{gathered} i\partial_t u - \Delta u + Vu = F, \quad (t, x) \in \mathbb{R}_t \times \mathbb{R}^3_x, \\ u(0, x) = f(x), \end{gathered} \tag{2.15}$$

*satisfies the estimate*

$$\|u\|_{L^p(\mathbb{R}_t; L^{(q,2)}_x)} + \|u\|_{C(\mathbb{R}_t; L^2)} \leq C(\|F\|_{L^{\tilde{p}'}(\mathbb{R}_t; L^{(\tilde{q}',2)}_x)} + \|f\|_{L^2}), \tag{2.16}$$

*where $\tilde{q}'$ is such that $1/\tilde{q} + 1/\tilde{q}' = 1$.*

Another direction of research, where Iordan also participated together with Angel Ivanov and Vladimir Georgiev, concerned the two-dimensional equivariant wave map problem with inhomogeneous source term in special domains. The initial results are presented in [4]. The result shows that if the initial data are in a better space than the natural energy space, then the corresponding norm of the solution blows up as $t$ goes to 0.

After the PhD thesis defence of Angel Ivanov in Pisa, he returned to Sofia, Bulgaria, and took a position at Technical University of Sofia. The work on Strichartz estimates for potential perturbations was continued.

The new project started with Iordan and Angel was really challenging. More precisely, Iordan, Angel and Vladimir Georgiev started to study the Laplace operator perturbed by a potential $V = V(x)$, $x \in \mathbb{R}^n$

$$\Delta_V = \sum_{j=1}^{n} \partial_j \partial_j - V = \sum_{j=1}^{n} \partial_{x_j} \partial_{x_j} - V \tag{2.17}$$

and their goal was to study the dispersive properties of the corresponding wave equation

$$\begin{cases} \partial_{tt} u - \Delta_V u = F, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n \\ u(0, x) = f_0(x), \ \partial_t u(0, x) = f_1(x). \end{cases} \tag{2.18}$$

The main assumption concerning the potential $V$ is the following one.

**Assumption 2.1.** *There exists $C > 0$ and $p$, $2n/3 \le p \le n$, such that we have*

$$\sum_{k\in\mathbb{Z}} 2^{2k-nk/p} \|V(x)\|_{L^p_{\{|x|\sim 2^k\}}} \le C. \tag{2.19}$$

The following estimate was obtained.

**Theorem 2.3.** *If $n \ge 4$, then for any potential $V$ satisfying* Assumption 2.1 *there exists $C > 0$, such that for any $f_0, f_1 \in S(\mathbb{R}^n)$ and any $F(t,x) \in S(\mathbb{R} \times \mathbb{R}^n)$ the solution $u(t,x)$ to (2.18) satisfies the endpoint estimate*

$$\|u\|_{L^2_t L^{p_0}_x} \le C\|f_0\|_{\dot H^\gamma_x} + C\|f_1\|_{\dot H^{\gamma-1}_x} + C\|F\|_{L^2_t L^{p_0'}_x}, \tag{2.20}$$

*where $p_0 = 2(n-1)/(n-3)$ and*

$$\gamma = \frac{n-1}{2} - \frac{n}{p_0} = \frac{n+1}{2(n-1)}.$$

The work was not completed, since after an incident in Sofia Angel passed away in 2008. Angel's demise took away unexpectedly a young and very promising researcher, who was one of the key members of the group working on dispersive PDE at Technical University. His colleagues and friends, among which Iordan Iordanov, George Venkov, Mirko Tarulli and Vladimir Georgiev, were deeply saddened and shocked by the death of Angel.

Over the next years (between 2008 and 2019) Iordan and Vladimir Georgiev had various contacts on research topics. The focus of discussion was mathematical models in finance and especially the use of copulas in these models.

The above shows the extremely creative approach of Iordan, his deep mathematical preparation and ability to work with Master and PhD students, and stimulate in this way the progress and development of mathematical research in Bulgaria.

## 3. Economic modelling

Iordan's work on re-establishing the Faculty of Economics and Business Administration at Sofia University naturally piqued his curiosity about the mathematical toolkit used in economic modelling. This was initially motivated by attempts to build cutting-edge curricula in mathematics and quantitative methods for economists but soon spilled over to research questions. As an expert in partial differential equations, Iordan was underwhelmed by their use in the economic sciences. He was well aware of developments around the Black-Scholes option pricing approach but found the use of PDE tools there almost accidental, a by-product of a different construct. His starting point was that the natural use of PDEs would be to model phenomena evolving through both space and time. This stimulated his interest in spatial economics.

It quickly became apparent that spatial economics, while conceptually closer to Iordan's views, does not abound with examples of applications of PDEs. Instead, a central theme of the spatial economics literature in the late 1990s and early 2000s was the analysis of models featuring transportation costs and potentially other costs arising from geographical separation. This type of model inspired Iordan's early work in spatial economics in collaboration with Stoyan Stoyanov and Andrey Vassilev.

### 3.1. Strategic trade models

The first contributions to the analysis of spatial models came in the form of two linked papers [20,42]. They study the adjustment of prices towards an equilibrium in a model of trade between two regions, respectively in discrete-time and in continuous-time settings. Trade takes place under conditions of strategic interactions, taking as given output, financial resources and transportation costs.

More precisely, the consumer in region I (consumer I) exogenously receives money income $Y_1 > 0$ in each period and so does the consumer in region II (consumer II), who receives money income $Y_2 > 0$. For each period $t$, in region $i$ a fixed quantity $q_i > 0$ of a certain good is supplied at a price $p_{i,t}$. The consumers place orders for the desired quantities in each region, observing their budget constraints and incurring symmetric transportation costs $\rho > 0$ per unit of shipment from the "foreign" region.

For each period $t$ the above situation is modelled as a static noncooperative game of complete information. The orders placed by consumer I in regions I and II, respectively, are denoted by $\alpha$ and $\beta$. Similarly, $\gamma$ and $\delta$ stand for the orders of consumer II in regions I and II. In period $t$ consumer I's strategy space $S_1$ is determined by the budget constraint and the nonnegativity restrictions on the orders:

$$S_1 = \{\alpha p_{1,t} + \beta(p_{2,t} + \rho) \leq Y_1,\ \alpha, \beta \geq 0\}. \tag{3.1}$$

Consumer II's strategy space in period $t$ is

$$S_2 = \{\gamma(p_{1,t} + \rho) + \delta p_{2,t} \leq Y_2,\ \gamma, \delta \geq 0\}. \tag{3.2}$$

The payoff function for consumer I is given by

$$P_1(\alpha, \beta, \gamma, \delta) = \min(\alpha, q_1) + \min(\beta, q_2 - \min(\delta, q_2)) \tag{3.3}$$

and that for consumer II by

$$P_2(\alpha, \beta, \gamma, \delta) = \min(\gamma, q_1 - \min(\alpha, q_1)) + \min(\delta, q_2). \tag{3.4}$$

The motivation behind the above formulation of the payoff functions comes from a restriction in market access taking the form of full protection for the local consumer. More specifically, when total orders for the respective region exceed the quantity available, the order of the local consumer is executed first to the extent possible and then the remaining quantity, if any, is allocated to the consumer from the other region.

In discrete time, a price is adjusted downwards if the quantity available in the respective region has not been entirely consumed. A price is adjusted upwards if a part of the orders placed in the respective region has not been satisfied. In particular, prices evolve according to the equation

$$\frac{q_i - q_i^{\text{cons}}}{q_i} = -\frac{p_{i,t+1} - p_{i,t}}{p_{i,t}} \quad \text{or} \quad p_{i,t+1} q_i = p_{i,t} q_i^{\text{cons}} \tag{3.5}$$

when $q_i > q_i^{\text{cons}}$ and according to

$$\frac{Y_i^{\text{res}} - p_{i,t} q_i}{p_{i,t} q_i} = \frac{p_{i,t+1} - p_{i,t}}{p_{i,t}} \tag{3.6}$$

when $Y_i^{\text{res}} > p_{i,t} q_i$ with $q_i^{\text{cons}}$ denoting the total amount consumed in region $i$ and $Y_i^{\text{res}}$ standing for the part of the region $i$'s income left unspent.

In continuous time the counterpart of equations (3.5) and (3.6) is given by the differential equation system

$$\frac{\dot{p}_i}{p_i} = -\frac{q_i - q_i^{\text{cons}}}{q_i} + \frac{Y_i - Y_i^{\text{cons}}}{p_i q_i}, \quad i = 1, 2. \tag{3.7}$$

For the above setup, the existence of Nash equilibria is proved and selection rules that pinpoint a unique equilibrium are formulated for edge cases. The Nash equilibria in the model are classified based on the relations among quantities produced and financial resources. The different price paths that can be obtained are also analysed and classified to establish the various possibilities of obtaining a price equilibrium. It turns out that in certain cases the laws governing price dynamics in discrete time lead to a zero price in one of the regions, which can be interpreted as a breakdown of economic activity in the region. This pathology is ruled out in the case of continuous-time price dynamics.

The model from [20] and [42] is generalised in [28, 30, 31]. The main novelty is that local consumers enjoy only partial protection in terms of preferential access to the good on their home market. Additionally, the formulation is general and dispenses with certain symmetry assumptions, allowing for differences between regions in terms of parameters such as income, size of the local market supply, degree of protection and transportation costs.

The partial protection framework is formalised through the following payoff function for consumer 1:

$$\begin{aligned} P_1(\alpha, \beta, \gamma, \delta) = \min(\beta, (1 - \epsilon_2) q_2 + \max(0, \epsilon_2 q_2 - \delta)) \\ + \min(\alpha, \max(q_1 - \gamma, \min(\alpha, \epsilon_1 q_1))). \end{aligned} \tag{3.8}$$

Similarly, the payoff function for consumer 2 is

$$\begin{aligned} P_2(\alpha, \beta, \gamma, \delta) = \min(\gamma, (1 - \epsilon_1) q_1 + \max(0, \epsilon_1 q_1 - \alpha)) \\ + \min(\delta, \max(q_2 - \beta, \min(\delta, \epsilon_2 q_2))). \end{aligned} \tag{3.9}$$

The interpretation of (3.8) and (3.9) is that a fixed share $\epsilon_1 \in (0, 1]$ of the quantity $q_1$ is reserved for consumer 1 and, similarly, a share $\epsilon_2 \in (0, 1]$ of the quantity $q_2$ is preferentially available to consumer 2. Consumers have the right to buy the respective quantities $\epsilon_1 q_1$ and $\epsilon_2 q_2$ but are not obliged to do so. After the local consumer buys a part or all of the preferentially available quantity, the remaining quantity of the good is offered to the foreign consumer. In turn, the foreign consumer can purchase part or all of this remainder and, if there is anything left, it is again offered to the local consumer.

In the extended model price dynamics are studied in discrete time and consequently several versions of (3.5) and (3.6) are used to describe the price changes over time. For this model, the existence of Nash equilibria is proved and a classification of the players' best-reply functions and the obtained Nash equilibria is presented. Price dynamics are studied by means of numerical simulations. The results show that different price adjustment rules used in conjunction with different parameter sets can produce very diverse outcomes. Examples range from trivial one-period "jumps" of prices to a steady state value to much less regular behaviour such as transitional dynamics settling down on a steady state value or cyclical behaviour. Overall, the specification of the price adjustment rules is crucial for the type of dynamics obtained and the interpretation of the results.

### 3.2. Spatial relocation models

Iordan's quest for an economic model with "genuine space" continued and an initial formulation appeared around 2006. The model considered is in essence microeconomic and studies the dynamic consumption-saving behaviour of an individual who is allowed to move in a suitable economic space in search of better job opportunities. The introduction of an explicit economic space that provides an abstraction for the physical, geographic space where agents operate was a first step in an agenda aimed at constructing an aggregate description of an economy with both spatial and temporal dimensions.

One version of the formulation is the following [22,23]. There is a consumer who, given an initial location in space $x_0$ and asset level $a_0$, supplies inelastically a unit of labour in exchange for a location-dependent wage $w(x(t))$, and chooses consumption $c(t)$ and spatial location $x(t)$ over time. The consumer has a finite lifetime $T$ at the end of which a bequest in the form of assets is left. This bequest provides utility to the consumer. More precisely, for $\rho$, $r$, $\eta$, $\xi$ and $p$ – positive constants, and $\theta \in (0,1)$, the consumer's choices are described by the optimal control problem of maximizing with respect to $c(t), z(t) \in \Delta$ the objective functional

$$J(c(t), z(t)) := \int_0^T e^{-\rho t} \left( \frac{c(t)^{1-\theta}}{1-\theta} - \eta z^2(t) \right) \mathrm{d}t + e^{-\rho T} \frac{a(T)^{1-\theta}}{1-\theta} \tag{3.10}$$

subject to

$$\dot{a}(t) = ra(t) + w(x(t)) - pc(t) - \xi z^2(t), \tag{3.11}$$

$$\dot{x}(t) = z(t), \tag{3.12}$$

$$a(0) = a_0 \geq 0, \quad x(0) = x_0 \in [0,1],$$

where $a(t)$, $x(t)$ are the state variables, assumed to be absolutely continuous, and $c(t)$, $z(t)$ are the control variables. The set of admissible controls $\Delta$ consists of all pairs of functions $(c(t), z(t))$ which are measurable in $[0, T]$ and satisfy the conditions

$$0 \leq c(t) \leq C, \tag{3.13}$$

$$|z(t)| \leq Z, \tag{3.14}$$

$$a(T) \geq 0. \tag{3.15}$$

The constants $C$ and $Z$ satisfy appropriately defined conditions that are technical in nature but do not affect the economic analysis of the problem. The constant $\rho > 0$ is a time discount parameter and $\theta \in (0, 1)$ is the utility function parameter. The control $c(t)$ represents physical units of consumption and the control $z(t)$ governs the speed of relocation in space. Relocation in space brings about two type of consequences: subjective disutility due to habit formation with respect to the current location and monetary relocation costs. These are captured through the speed of movement in space $\dot{x}(t)$ or, equivalently, $z(t)$, transformed through a quadratic function. The parameters $\eta, \xi \geq 0$ multiplying this function measure respectively the subjective disutility from changing one's location in space and the relocation costs in monetary terms. The parameters $p > 0$ and $r > 0$ stand for the price of a unit of consumption and the interest rate, respectively.

The above optimal control problem presents a mathematical challenge, since its set of generalised speeds is not convex, which is a key element of standard proofs of existence of solutions. This situation requires a different approach to show existence, which is done in [21] for a generalised version of the problem (3.10)–(3.15) that dispenses with explicit functional forms and works with general convexity and concavity assumptions instead.

Further analysis looks at the basic case where economic space is represented by the real line or, more specifically, a subset of it, the interval $[0, 1]$. This is modelled by taking the initial location $x_0 \in [0, 1]$ and requiring the location-dependent wage, which is positive in $(0, 1)$, to be negative outside $[0, 1]$ (as a way of circumventing the need to introduce state constraints in the model) and to satisfy additional assumptions. Necessary conditions for optimality based on Pontryagin's maximum principle are derived and, based on them, various model properties (e.g., bounds on terminal assets $a(T)$) are established. The implications of particular functional forms for the wage distribution $w(x)$ are investigated. A first set of results confirms that for a single wage maximum economically intuitive behaviour can be obtained, namely, moving towards regions where higher wages can be earned. Further results for the case of a wage distribution with two local maxima, included in Andrey Vassilev's PhD dissertation, reveal more complex behaviour, including the possibility of moving towards the smaller local maximum depending on the length of the planning horizon, as well as the possibility of having multiple solutions.

The paper [1] further extends this line of research by looking at the case of two-dimensional economic space. This work provides results on existence and necessary conditions that are the counterpart of the ones obtained in [23]. The analysis also reveals the limitations of using a negative wage as a substitute for state constraints for two-dimensional domains of more complex (e.g., non-convex) shape. In particular, it turns out that in two-dimensional economic space it is possible to have an optimal trajectory in the spatial variables that passes through a region with negative wage values in order to reach faster a region with high positive wage values. Numerical investigation of the model shows that even in the simplest case of a two-dimensional wage distribution with a single maximum intuitive but nontrivial results can be obtained. More specifically, depending on the exact form of the wage distribution, the agent may have incentives to relocate in economic space along various nonlinear trajectories. This finding suggests that for more complicated two-dimensional wage distributions significantly more complex types of behaviour can be expected.

### 3.3. Models of small open economies

Iordan Iordanov also worked on models that help understand macroeconomic developments in small open economies. These models can be classified as belonging to the dynamic stochastic general equilibrium strand of the economic literature. A first attempt to develop such a model is documented in [24], which modifies a standard model of a small open economy to fit the case of an economy operating under a currency board arrangement. The model is tailored to the Bulgarian economy both in terms of how the currency board is introduced and also in terms of its calibration to data and empirical facts. It also specifies the government behaviour in a way that is consistent with its prominent role in situations where monetary policy is passive. Model properties are illustrated by studying dynamics under three scenarios: a shock in the prices of energy resources, a shock in the productivity of the non-tradable sector and a shock in tradable goods prices. The authors conclude that the responses of the model variables with respect to various shocks that affect the economy are plausible and correspond to usual consequences entailed by the main economic mechanisms built into the model but further work is needed to improve the specification of the government sector and introduce nominal rigidities and adjustment costs in the analytical framework.

The directions for model improvement identified in [24] are tackled in [26]. In this work the government is modelled as following a set of fiscal rules that aim to stabilise output fluctuations while maintaining fiscal stability. This approach more closely corresponds to guiding principles and best practices for conducting fiscal policy. Additionally, it reflects public discussions on fiscal policy principles that were ongoing in Bulgaria at the time of writing. The paper also improves on the framework adopted in [26] by introducing monopolistic competition and Calvo pricing to account for nominal rigidities and adjustment costs. The simulation results demonstrate the potential of using fiscal rules to stabilise the economy when monetary policy is constrained because of the currency board arrangement.

A final contribution in the modelling of small open economies appeared in [27]. This work suggests a way of extending the popular Galí-Monacelli (G-M) model to the case of a world economy consisting of several regions, interpreted as monetary unions. The paper provides the counterparts of selected derivations from the G-M model which lead to a representation of domestic output as a function of domestic and foreign prices, and exchange rates. It also develops an alternative index form for aggregators that yields qualitatively similar results to the usual one with the added benefit of having explicit dimensionality consistency.

### 3.4. Educational contributions

Besides his purely research-oriented contributions to economic modelling, Iordan was always interested in pushing the boundary where the tools for conducting research and presenting economic models are concerned. One of his pet peeves was with the way many economics papers present results by providing a model formulation and then stating the final form of the optimality conditions describing the purported solution without any reference to the method for deriving these conditions. Frequently in such cases reference is made to a Lagrangian formalism but, if the derivation is shown at all, it is little more than the mechanical application of a recipe. More precise formulations and justifications exist but are scattered throughout the literature. To partially remedy this, the paper [25] collects and presents some results on necessary conditions and sufficient conditions for optimality in dynamic optimization problems with explicit controls when one approaches the solution through a Lagrangian formalism.

Other contributions to the teaching of mathematical methods in economics appear in [29] and [32]. Both follow an agenda outlined in the preface to [44], where the role of using numerical methods and computer simulations to study economic models is emphasised. The paper [29] offers an approach to deepening the students' understanding of the Solow growth model by presenting an interactive software application that helps students get acquainted with the main properties of the model. The application is implemented in Python on the basis of the IPython kernel for the Jupyter Notebook. An advantage of this approach is that it hides the implementation details from the typical user while still allowing advanced students to access the application internals and learn from the underlying program code.

The paper [32] is similar in spirit to [29] in that it employs the same interactive approach and computational framework in order to study the popular IS-LM model.

## 4. Modelling in finance and insurance

Iordan Iordanov also researched problems related to modelling applications in finance and insurance. His papers in this field are focused on copula functions and are co-authored with his PhD students Nikolay Chervenov and Boyan Kostadinov.

Copulas appear as a modern tool in creating flexible probability models with more than one random variable. Any multivariate model could be constructed by means of copulas. They are widely applied in fields such as insurance, finance, or any field where there are interacting processes (risks) and there is a need to measure their dependence. Iordanov's works on copulas were innovative and unlike the most papers related to copulas, where considerations are restricted to functions which are well-defined in each point and benefit from copulas being Lipschitz functions on their domain, he and his co-authors considered copulas under more general assumptions.

The paper [17] deals with 2-dimensional copula functions and proposes a method to simplify the otherwise complicated task of verifying the 2-increasing of $C$-volume of the copula by using simple differentiation, followed by a method to obtain a class of copulas as a solution of a boundary value problem in a suitable Sobolev space.

Precisely, a weakly 2-increasing function is defined by using weak derivatives (derivatives in the sense of distribution theory), with $(H_{xy}, \varphi) \geq 0$, for any test function $\varphi \geq 0$ in $\mathscr{D}(\mathbb{R}^2)$. It is proved that, for $H \in \mathscr{D}'(\mathbb{R}^2) \cap C^0(\mathbb{R}^2)$, the standard 2-increasing and weakly 2-increasing properties are equivalent.

Furthermore, despite some restrictions imposed by Sobolev spaces, in this work a family of copulas $C$ is constructed based on a given probability density $\partial_{uv}C$, using the concept of weak derivative. The main result is solving the boundary value problem

$$\begin{aligned}
&\partial_{uv}C(u,v) = f(u,v) \text{ in } I^2 = [0,1] \times [0,1] \text{ (in weak sense);}\\
&C(u,0) = 0 = C(0,v);\\
&C(u,1) = u,\ C(1,v) = v \text{ for all } u,v \in I = [0,1],
\end{aligned}$$

under certain conditions on $f$. First the case when $f$ is smooth function is considered and later it is generalized to $f \in W^{-1,p}(I^2)$, $p > 2$. One might initially think of this problem as a Dirichlet problem for the wave equation, but it turns out to be a Goursat problem for hyperbolic equations. However, it is important to note that this is an ill-posed boundary value problem.

While in [17] the results are given rather schematically, in [18] detailed proofs and numerous examples are given for the same problems. The paper pays special attention to the *a priori* estimate, which ensures the uniqueness of the solution $C$. Namely, there exists a constant $M$, which does not depend on $f$, such that $\|C\|_{W^{1,p}(I^2)} \leq M\|f\|_{L_p(I^2)}$. Next, the work provides detailed proofs of theorems on the existence of the solution in both the smooth $f \in L_p(I^2)$ and generalized $f \in W^{-1,p}(I^2)$, $p > 2$, cases.

Further works extend the 2-dimensional case to the $n$-dimensional, and study the $n$-increasing property, existence and uniqueness of the solution of the boundary value problem when $n > 2$.

The paper [2] defines the Goursat problem over the unit cube $I^n$. It is proved that there exists a unique solution of the problem

$$(-1)^n(C, \varphi_{x_1,\ldots,x_n}) = (f, \varphi)$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n \setminus \bigcup S_j)$ restricted over $I^n$, where $S_j = I^n \bigcap \{(u_1,\ldots,u_n) \in \mathbb{R}^n \mid u_j = 1\}$ are the sides of $I^n$ passing through the vertex $(1,\ldots,1)$ and the function $C$ satisfies the usual boundary conditions for a copula function $C(u_1,\ldots,u_n) = 0$, if at least one of the coordinates $u_1,\ldots,u_n$ equals zero.

The article [3] was Iordanov's final work on the topic. The paper presents two generalizations of the concept of an $n$-increasing function. The main result on the subject states that a function $H \in W^{1,p}(I^n)$ is weakly $n$-increasing in $I^n$ if the inequality

$$(-1)^n (H, f_{x_1,\ldots,x_n}) \geq 0$$

is fulfilled for all $f \geq 0$ in $W^{n-1,q}(I^n)$ such that $f$ and its derivatives vanish on the sides of $I^n$ passing through the vertex $(1, \ldots, 1)$.

This makes it easy to obtain several important results for copulas. In particular, the authors apply their approach to the bivariate Archimedean copulas, which are a special class of copulas. The proof of the characteristic property for Archimedean copula is completely new in the literature.

Another problem studied in [3] relates to the construction of $n$-dimensional copulas by using values of their derivatives $\frac{\partial^n}{\partial x_1 \ldots \partial x_n}$. To this end, the paper uses the generalized concept of an $n$-increasing function and expands the results of a version of the Goursat problem in $n$-dimensional cube in $\mathbb{R}^n$ proved in [2]. It is worth mentioning that the local conditions required to solve the boundary value problem in the 2-dimensional case are equivalent to those imposed in the $n$-dimensional case.

A notable application of the results presented in this section can be found in Nikolay Chervenov's PhD dissertation. The work considers a real-life problem for insurance risk assessment and solves it with numerical methods. The desired copula was constructed using the approach of Iordanov and Chervenov, and was obtained as a solution of a boundary value problem using real data from a Bulgarian insurance company.

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