ГОДИШНИК

НА

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„СВ. КЛИМЕНТ ОХРИДСКИ“

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Aims and Scope. The *Annuaire* is the oldest Bulgarian journal, founded in 1904, devoted to pure and applied mathematics, mechanics and computer sciences. It is reviewed by *Zentralblatt für Mathematik*, *Mathematical Reviews* and the Russian *Referativnii Jurnal*. The *Annuaire* publishes significant and original research papers of authors both from Bulgaria and abroad in some selected areas that comply with the traditional scientific interests of the Faculty of Mathematics and Informatics at the "St. Kliment Ohridski" University of Sofia, i.e., algebra, geometry and topology, analysis, mathematical logic, theory of approximations, numerical methods, computer sciences, classical, fluid and solid mechanics, and their fundamental applications.

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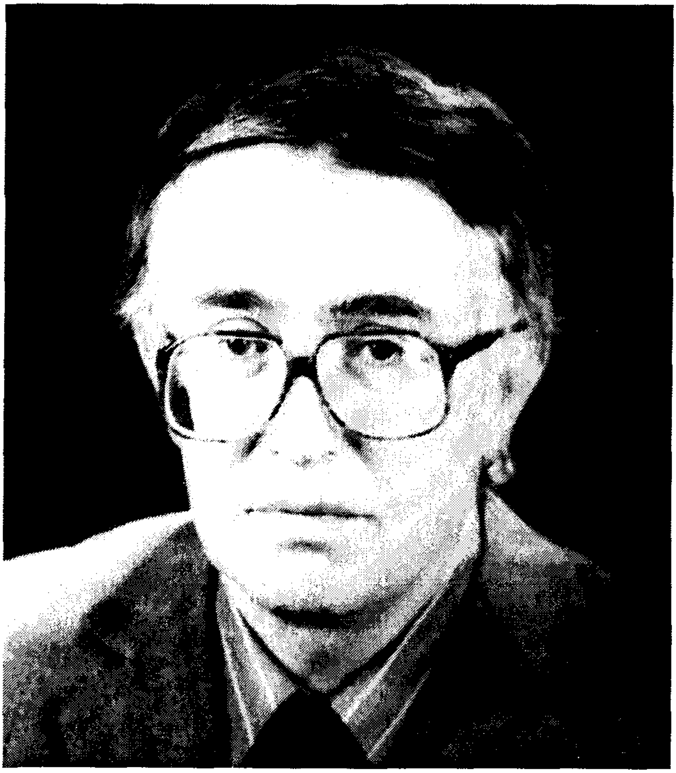
IN MEMORIAM



Проф. дмн Константин Здравков Марков

1945 - 2003

На 7 август 2003 г. почина Константин Здравков Марков, доктор на математическите науки, професор по механика на деформируемото твърдо тяло във Факултета по математика и информатика на Софийския университет „Св. Климент Охридски“, ръководител на катедра „Механика на непрекъснатите среди“, главен редактор на Годишника на СУ - ФМИ. Нека кратките биографични данни, поместени по-долу, върнат още веднъж светлия спомен за него.

Проф. дмн Константин Марков е роден на 28 март 1945 г. в София. През

1963 г. завършва средното си образование в софийската 22-ра гимназия с изграден силен интерес към математиката, на която твърдо решава да се посвети. През същата година, след издържан успешно конкурс, става студент в Математико-механическия факултет на Санкт Петербургския (тогава Ленинградски) университет. С това желанието му да се обучава в световно призната математическа школа е изпълнено, но не напълно, защото съществуващият и до днес странен феномен „държавна поръчка“ му дава възможност да кандидатства само за специалността механика, но не и за предпочитаната тогава от него специалност математика. Решението, което той намира за себе си, е колкото елементарно, толкова и нетривиално за изпълнение. А то е да се обучава едновременно и по двете специалности. Това свое решение той довежда до впечатляващ край. През 1968 г. се дипломира с отличие и по двете специалности със защити на дипломни работи в катедрите „Математическа физика“ (интерполация на линейни оператори) и „Теория на еластичността“ (моделиране на анизотропни тела). Получава препоръка да продължи с редовна аспирантура (1969-1972 г.) обучението си в катедра „Теория на еластичността“. След успешна защита на дисертация на тема „Теоретико-групов анализ на определящите уравнения за нелинейни анизотропни среди с пълзене“ на К. Марков е присъдена научната степен „доктор по математика“ (тогава „кандидат на физико-математическите науки“). През осемте години на следване и аспирантура, години на преплитане на чиста и приложна математика, у К. Марков се формира по естествен път вкусът към живата математика, т.е. към приложенията на математиката в изследването на реални среди и процеси или, иначе казано, към това, което днес най-общо се определя като математическо моделиране. На този свой вкус той остава верен и в изследванията си, които му донасят широко международно признание, и в дейността си на преподавател, с която трайно привлича много свои ученици за каузата на математическото моделиране. На всички, които се интересуват от многобройните приложения на математиката в различните клонове на човешкото познание, К. Марков направи неотдавна чудесен подарък, като подготви, превъзмогвайки проблемите, свързани с тежкото му заболяване, монографията „Математическо моделиране“, издадена от Университетско издателство „Св. Климент Охридски“ през 2002 г. Само година по-късно последва втори подарък за онези, които проявяват интерес към механоматематическото моделиране, т.е. към специфичните приложения на принципите и методите на математическото моделиране по отношение на изследването на механичното поведение на твърдите деформируеми тела и на флуидите. Този втори подарък е монографията „Механика на непрекъснатите среди“, издадена през 2003 г. от същото университетско издателство.

Професионалната кариера на К. Марков започва в сектора „Механика на непрекъснатите среди“ към Института по математика и механика - БАН, където той е научен сътрудник от 1972 до 1977 г. През периода от 1977 до 1989 г. е доцент, а от 1989 г. – професор по механика на деформируемото твърдо тяло във Факултета по математика и информатика при СУ „Св. Климент Охридски“. След успешната защита през 1983 г. на дисертация на тема „Механоматематическо моделиране на микронееднородни деформируеми тела“

пред Специализирания научен съвет по механика при ВАК на К. Марков е присъдена научната степен доктор на математическите науки.

Впечатляваща е научната продукция на К. Марков. Той е автор на повече от 70 научни труда, публикувани освен във водещите наши научни списания, такива като *Доклади на БАН*, *Годишник на Софийския университет*, *Mathematica Balkanica*, *Теоретична и приложна механика - БАН* и др., така и в най-престижните в областта на приложната математика и математическото моделиране чуждестранни научни журнали, сред които *Proc. Royal Society London - Ser. A*, *J. Mechanics and Physics of Solids*, *Int. J. Solids and Structures*, *SIAM J. Applied Mathematics*, *ZAMM*, *IMA J. Applied Mathematics* и много други. Тази научна продукция отразява резултатите от изследванията на К. Марков по широк кръг от проблеми на механоматематическото моделиране на среди с разнообразна структура и с различни типове поведение. Това са изотропни и анизотропни, обратимо (еластично) и необратимо (пластично и вискозно) деформируеми среди както с еднородна, така и с нееднородна, детерминирана или случайна вътрешна структура. Особено значими са постиженията на К. Марков от последните години, свързани с адекватното моделиране на среди със случайна, при това неизвестна, начална или придобита (например поради микроразрушаване, съпровождащо деформационните процеси) дисперсна вътрешна структура. Трудовете му, посветени на този кръг от въпроси, демонстрират силата на развития от него и признат от изследователите в тази област подход, в рамките на който, въз основа на построяване на асимптотично-точни решения за параметрите на случайните полета, характерни за микроструктурата на такива среди, се извличат вариационни оценки за макроскопичните механични свойства на същите среди. Освен с чисто научния си принос тези трудове привличат вниманието и с демонстрираната в тях комбинация от впечатляваща математическа ерудиция и завидна математическа сръчност. С тази комбинация от ерудиция и сръчност, характерна за цялостното му научно творчество, К. Марков изгради завиден авторитет и спечели уважението на много от своите колеги у нас и в чужбина.

Особено отговорно беше отношението на К. Марков към ангажиментите му на преподавател. Основна негова грижа винаги е била да излага четения материал по възможно най-достъпен за студентите начин, да илюстрира този материал с реални примери и с това да им помогне да осмислят и „почувстват“ неизбежните в математиката абстрактни понятия. Освен лекциите по Аналитична механика, Математическо моделиране, Математични методи в механиката, Механика на непрекъснатите среди, Механика на композиционните материали, Механика на увреждането, които чете повече от 30 години във ФМИ, К. Марков е чел лекции още като аспирант в Санкт Петербургския университет (Тензорна алгебра и тензорен анализ, 1970–1972 г.), като гост–преподавател в Шуменския университет (Математически анализ и Аналитична механика, 1981-1992 г.), в Института по химически и биотехнологии в Разград (Техническа механика, Съпротивление на материалите, 1994-1996 г.), в Истанбулския технически университет (Механика на повреждането и Механика на композиционните материали, 1992 и 1994 г.). Подготвил е десетки дипломанти във

ФМИ при СУ и в Шуменския университет, както и двама доктори по математика.

К. Марков е участвал като желан гост в редица престижни международни научни форуми и е бил организатор на немалко от тях. Особено представителни са участията му с доклади в международни конференции и конгреси, организирани от Нютоновия институт по математически науки, Кембридж, Великобритания (1999 г.), Математическия институт в Оберволфах, Германия (1983, 1986, 1990 г.), Националната научна фондация на САЩ, Вашингтон (1993 г.), от университетите в Солун, Гърция (1990, 1993, 1998 г.), Пиза, Италия (1997 г.), Ливърпул, Великобритания (1999 г.), от университети и научни институти във Франция, Полша, Швеция и др. Сред научните форуми, на които К. Марков е организатор и съорганизатор, са българските Национални конгреси по теоретична и приложна механика (Варна, 1977 и 1981 г.), Българо-гръцкия симпозиум по математическо моделиране в механиката и техниката (Гьолечица, 1989 г.), симпозиумът Евромех 278 по микроструктура и ефективни свойства на композиционни материали със случайна структура (Шумен, 1991 г.), осмата и деветата международни конференции по континуални модели и дискретни среди (Варна, 1995 г., и Истанбул, 1998 г.). Като гост-изследовател К. Марков работи по съвместни проекти в Департамента по инженерни науки на Истанбулския технически университет, в Департамента по математически науки в Бат, Великобритания, в Департамента по математика при Университета в Торино, Италия.

С авторитета си сред чуждите изследователи, активно работещи в неговата област, К. Марков става търсен научен редактор на специализирани сборници с научни трудове в областта на математическото моделиране. Такива са Recent Advances in Mathematical Modelling of Composite Materials (World Scientific, 1994), Heterogeneous Media: Modelling and Simulation (съвместно с L. Preziosi, Birkhäuser, 1999), както и сборници с материали на проведени у нас и в чужбина конгреси и конференции. От 1997 г. той е регионален редактор на *Zentralblatt für Mathematik*.

Качествата му на учен, преподавател и просто на колега и човек с доказано отговорно отношение към ангажиментите, които поема, са обективната причина К. Марков да бъде постоянно привличан в работата на свързаните с управлението на науката у нас институции. Той е научен секретар (1979–1992 г.) и председател (1993 - 1998 г.) на Специализирания научен съвет по математика, информатика и механика при ВАК, член на Комисията по математически науки при ВАК (1990–1992 г. и след 1998 г.), член на Научно - експертната комисия по математика и механика към Националния фонд „Научни изследвания“ (1991–1992 г.), член на Управителния съвет (1993–1999 г.) и на Изпълнителното бюро (от 1999 г.) на Националния съвет „Научни изследвания“. От 1999 г. К. Марков е член и на Постоянната комисия по природни науки към Акредитационния съвет на НАОА.

В продължение на повече от 20 години К. Марков вложи много усилия за осъвременяването на Годишника на СУ - ФМИ и утвърждаването на авторитета му. От 1980 г. той е член на редакционната колегия на Годишника, а от 1996 г. е негов главен редактор. В тази си роля той пое върху себе си не само голям обем организационна работа, но извърши лично и значителна, особено полезна чисто техническа работа, свързана с предпечатната подготовка на отделните томове и с тяхното външно оформление.

Колегията на ФМИ - СУ ще запази добър спомен за проф. дмн Константин Марков като за ерудиран и предан на работата си учен и преподавател, добър колега и светъл човек.

Поклон пред паметта му!

Факултетен съвет - ФМИ
Ред. колегия на Годишник на СУ - ФМИ
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SOME SHORT HISTORICAL NOTES ON DEVELOPMENT OF MATHEMATICAL LOGIC IN SOFIA

DIMITER SKORDEV

We present some information about the pre-history and the history of development of mathematical logic in Sofia. The history of the so-called Sector of Mathematical Logic (existing from 1972 to 1989) is considered in some detail.

Keywords: mathematical logic, history, Department of Mathematical Logic, Sofia University, Faculty of Mathematics, Bulgarian Academy of Sciences, Institute of Mathematics

2000 MSC: main 01A60, secondary 03-03

Currently, mathematical logic in Bulgaria has some presence not only at Sofia, but also at several university centres. However, I shall restrict myself only to its history in Sofia, since both the history and the present state of the field in those other places are far from being as abundant as in Sofia. In addition, I shall speak mainly about the earlier part of the history, since it is probably the less known to the audience. The year 1989 will be regarded as the end of that period of time. Besides, I shall actually speak mostly about the history of the Department of Mathematical Logic, meaning the former Sector of Mathematical Logic and the two currently existing units that succeeded it in 1989. In fact, almost all people who work or have worked in mathematical logic at the Sofia either are present or former members of this department, or have graduated from it. There are only a few exceptions. Bojan Petkanchin (1907–1987), a greatly respected professor in geometry at the Sofia University, is one of them, and his pioneering role in the history of mathematical logic in Sofia will be considered further. Another exception is Nadejda Georgieva (1931–1995) – her education at the Sofia University was completed before the time of mathematical logic had come there, she was a professor at the Institute of Mining Engineering and taught General Mathematics, but she did research mainly in mathematical logic after a specialization abroad. (Information about some other people not mentioned here can be found in Section 3 of [15].)

As well known, the foundations of mathematics have been and still are an important background and an object of study for mathematical logic. The interest in them has a long tradition in Sofia. For example, several competently written articles on the foundations of arithmetic were published in the Journal of the Physico-Mathematical Society in Sofia almost a century ago, in particular, a series of articles by an author publishing under the alias "Uni". (Corresponding references can be found in the survey [16]). Mention may be made here also of a lecture of the German mathematician Otto Blumenthal (1876–1944), which was held in Sofia in 1935. The title of this lecture is "The life and the scientific work of David Hilbert", and the contents of the lecture is known from its Bulgarian translation [1]. A short description of Hilbert's work on the foundations of arithmetic and logic can be found on pp. 49-50 there, namely, the idea of Hilbert's program is briefly explained (without using the term "mathematical logic" and, unfortunately, keeping off the then already known problems encountered by that program).

One may be curious about the earliest occasions when mathematical logic was explicitly mentioned in Sofia in public. The first such occasion, known to me, is a lecture held in 1945 by the Bulgarian mathematician Yaroslav Tagamlitzki (1917–1983). An information about it can be found in [2], where one sees a list of 11 titles of talks given in the Sofia University during the summer semester of the academic year 1944/1945. Tagamlitzki's lecture is the second one in the list and has the title "On some problems of mathematical logic". To my regret, there is no other information about the contents of this lecture. I should like to note that Tagamlitzki has been an assistant professor in 1945 and one of the eminent professors in the Sofia University later. It is known that he has attended a series of Blumenthal's lectures still being a high school student, so probably he has attended also the above-mentioned one.

The next ten years after 1945 have not been favourable for doing mathematical logic in Bulgaria. As in the former Soviet Union at that time, some scientific areas became practically forbidden then in our country too. Although mathematical logic did not completely fall in such a position, it was definitely not favoured by the political authorities, since only the so-called dialectical logic was officially accepted by them. Nevertheless, several mathematical courses at Sofia University have been taught in a modern logically clear way that raised the interest towards logical problems and prepared a ground for further acquaintance with their contemporary treatment. I would like to mention as examples of such courses the ones in Analysis, taught by Professor Y. Tagamlitzki, and in Foundations of Mathematics, taught by Professor B. Petkanchin. The second one of them was especially helpful in this respect thanks to its subject matter and the irreproachable way of its presentation. (The corresponding monumental textbook [3] has some of the features of a monograph).

At a certain moment of time prior to 1960 the authorities in the Soviet Union changed their attitude to mathematical logic. (This happened when they learned about the importance of electronic computing machinery and about a relation of mathematical logic to it). When those authorities finally admitted that mathematical logic should be considered as a legitimate and promising part of mathematics, so did the Bulgarian ones.[1] An attempt to take advantage of the changed situation was made in July 1959 in a plan signed by the Director of the Institute of Mathematics[2] at the Bulgarian Academy of Sciences. The plan contained a clause about having next year a Bulgarian Ph.D. student in mathematical logic in the German Democratic Republic or Poland. (However, as far as I know, this clause was not put into effect.) Several months later, in November 1959, a decree of the Central Committee of the Bulgarian Communist Party and of the Ministerial Council planned a number of activities aiming at the scientific progress of Bulgaria. The decree assigned many tasks to the Academy of Sciences, one of them being the creation of seven sections at the Institute of Mathematics, including a Section of Mathematical Logic (cf. [4, p. 2]). Such a section was eventually created, and Professor B. Petkanchin became its head. (Already in 1960 some documents listed this section in the structure of the Institute of Mathematics, but with an empty set of regular members). The section existed until the end of 1970, remaining a small one during all this time.

The first course of mathematical logic in the Sofia University was given by Professor B. Petkanchin during the academic year 1959/1960. It contained material from propositional and predicate calculus that is usually present in such courses, including also Gödel's Completeness Theorem. Starting from 1962 (soon after my return back to Bulgaria from a one-academic-year stay at the Department of Mathematical Logic of the Moscow State University) I also began giving some lecture courses in the field of mathematical logic, especially a course in Recursive Function Theory.

No particular department at the Sofia University was officially engaged in teaching mathematical logic at that time. Me and the somewhat younger colleagues Petio Petkov and Dimiter Vakarelov had positions in the Faculty of Mathematics at various departments, whose main teaching duties had been in other scientific areas. The research in mathematical logic was something additional to our main obligations too, with the following two exceptions: for P. Petkov in a several-years period (ending in 1970), when he was a Ph.D. student at the Department of Mathematical Logic of the Moscow State University, and for me during the already mentioned stay there and during a second one in the academic year 1968/1969. At the end of 1970 and the beginning of 1971 the so-called United Centre of

[1] On the other hand, the official tradition to deny or undervalue the philosophical significance of mathematical logic continued during a much longer period of time, as seen for example from [5]. Even now some after-effects of this can be still observed.

[2] The names "Institute of Mathematics" and "Faculty of Mathematics" (to be encountered further) are used in these notes as the invariant parts of two names that changed in the course of time. For example, an addition "and Mechanics" was present in them during a certain period of time, and this addition was replaced by "and Informatics" later.

Mathematics and Mechanics was formed. It encompassed the Faculty of Mathematics of the Sofia University and the Institute of Mathematics of the Bulgarian Academy of Sciences. The structure of the United Centre included a unit called Sector of Topology and Mathematical Logic. Professor Doitchin Doitchinov (1926–1996), one of the Bulgarian topologists, was appointed chief of the sector. (To be more precise, I ought to mention that all chiefs of such units figured during a long period of time as temporary ones in the administrative hierarchy of the United Centre, but their duties were not substantially influenced by this.) The staff of the Sector of Topology and Mathematical Logic consisted of specialists in both scientific areas from the Faculty of Mathematics and from the Institute of Mathematics. Before the formation of the United Centre, these people had belonged to different departments of the faculty and sections of the institute.

The logicians, who became members of the sector at the beginning, are the following ones (in alphabetic order of surnames): Radoslav Pavlov, Petio Petkov, Dimiter Skordev, Vladimir Sotirov and Dimiter Vakarelov. All of them except Pavlov had graduated from the Faculty of Mathematics of the Sofia University, and Pavlov had graduated from the Moscow State University not long ago. Petkov, Skordev and Vakarelov themselves belonged to the staff of the faculty at the time of the formation of the sector, and they had been members of the departments of Applied Mathematics, of Analysis and of Geometry, respectively. On the other hand, Sotirov and Pavlov were members of the Section of Mathematical Logic at the Institute of Mathematics at that time (Sotirov entered the staff of the section in January 1970 and Pavlov entered it in August 1970). In 1971 one more logician came into the sector – the late Georgi Gargov (1947–1996), who had then just graduated from the Moscow State University. However, in October 1972 Gargov went to Moscow again and became a Ph.D. student there. (He returned to his work in Sofia only several years later).

In 1971 among the logicians in the sector only Petkov and Skordev had Ph.D. degrees, and Skordev was also an associate professor. (Petkov received his Ph.D. degree from the Moscow University in 1970 after defending a dissertation in constructive mathematical logic; Skordev's Ph.D. degree was received three years earlier, but it was from the Sofia University – for some results in functional analysis.) It is appropriate to mention that in the course of the next eight years the other three of the five people listed above also received Ph.D. degrees – Vakarelov received his one from the Warsaw University, and Pavlov and Sotirov – from the Moscow University.

The heterogeneity of the Sector of Topology and Mathematical Logic was fairly obvious – actually both its components, the topological and the logical one, had quite good activities in their fields, but without any substantial interaction between them. The main areas in mathematical logic developed by members of the logical part were recursive function theory, constructive mathematical logic, many-valued logic, algorithmic problems in algebra, set theory.

A remarkable event in the life of the logical part of the sector took place in the autumn of 1972, namely a one-month visit of the eminent Russian logician Andrei

Andreevich Markov (1903–1979), head of the Department of Mathematical Logic in the Moscow State University. On the 6th of October Markov delivered a lecture under the title "Introduction into constructive mathematical logic", and nine other lectures on constructive mathematical logic followed it during the next three weeks.

Having in mind the heterogeneity of the sector, its chief took reasonable steps toward a correction of this situation. In a report dated March 7, 1972, Professor Doitchinov suggested to the authorities of the United Centre to split the sector in two – a Sector of Topology and a Sector of Mathematical Logic. He indicated that there were sufficiently many specialists in the sector (7 in topology and 6 in mathematical logic) for the normal functioning of the two prospective sectors. The splitting became reality not later than a few days after the departure of Markov from Sofia. The decision was taken by the Bulgarian Academy of Sciences on the 8th of November, 1972, and on the 20th of November an order of the Rector of the Sofia University and of the Head of the Academy appointed Dimiter Skordev as chief of the Sector of Mathematical Logic.

No essential changes occurred in the activity of the logic group after that administrative change. In fact, few things of administrative nature depended on the newly appointed chief, who never became member of the Communist Party. Fortunately, all members of the group had the abilities and the enthusiasm needed for fruitful work in the field of mathematical logic – both in research and in teaching, and the work did not get embarrassed by careeristic conflicts inside the group. In addition, there were certain opportunities for rising our research qualification by specializations in leading universities and other scientific institutions abroad, as well as by contacts with distinguished foreign logicians who visited our group. I shall list some of the first realizations of these opportunities.

Dimiter Vakarelov had a 7-month specialization in the Warsaw University during the academic year 1972/1973. Radoslav Pavlov and Vladimir Sotirov entered external Ph.D. study at the Moscow University in 1973. Dimiter Skordev had a 3-month specialization in the USA during the period November 1974 – February 1975. (He stayed at the Stanford University for about two months, and the rest of the time was spent mainly at UCLA.) Professor Helena Rasiowa (1917–1994) from the Warsaw University visited the sector for about a month in the academic year 1974/1975 and she delivered nine lectures on algorithmic logic in the period from February 17 to March 11, 1975. I would like to mention that professor Rasiowa was a guest of the sector many times later and gave a lot of other talks to its members and to a broader audience. A great number of other specialists in mathematical logic and in related fields from abroad visited the sector and gave talks in the next years of its existence too, many of them several times. Here is a possibly incomplete list of them (37 persons, listed in alphabetic order of surnames): S. N. Artemov (Moscow), F. G. Asenjo (Pittsburg), N. da Costa (Sao Paolo), B. Dahn (Berlin), A. G. Dragalin (Debrecen), K. Dyrda (Kelce, Poland), A. P. Ershov (Novosibirsk), Ju. L. Ershov (Novosibirsk), J. Gehne (Berlin), K. Härtig (Berlin), V. E. Itkin (Novosibirsk), I. Janicka-Żuk (Kelce, Poland), A. Jankowski (Warsaw), M. J. Kanovich (Kalinin), A. Kučera (Prague), L. L. Maksimova (Novosibirsk), I. A. Malcev (Novosibirsk),

A. Mazurkiewicz (Warsaw), N. N. Nepeyvoda (Izhevsk), V. A. Nepomnyashchy (Novosibirsk), E. Orlowska (Warsaw), G. Priest (Australia), C. Rauszer (Warsaw), V. Rybakov (Krasnoyarsk), L. Rudak (Warsaw), A. Salwicki (Warsaw), A. L. Semyonov (Moscow), A. Shen (Moscow), N. Shilov (Novosibirsk), D. I. Sviridenko (Novosibirsk), A. Trybulec (Bialystok), I. Urbas (Australia), P. Urzyczyn (Warsaw), V. A. Uspensky (Moscow), J. van Benthem (Amsterdam), M. Weese (Berlin), G. Wolf (Berlin). This list does not include some names of people who gave talks at conferences organized by the sector (these conferences will be considered later in this report).

The subject matter of the research done in the sector gradually became wider. In particular, more attention was directed towards the interconnections with theoretical computer science. This process was also accelerated to some extent by the reinforcement of the sector with several new people in the course of time. In January 1976 Georgi Gargov came back to Sofia from his Ph.D. study and entered the staff of the sector (two months later he got his Ph.D. degree in Moscow). A completely new person came in the Sector of Mathematical Logic in July 1976 – then Anatoly Buda moved from the Sector of Software Programming after finding out that our research field was better related to the directions of computer science studied by him in Novosibirsk (Buda had received his education from the Novosibirsk State University, there he had defended a Ph.D. dissertation under the supervision of A. P. Ershov in 1975). Three other persons came later – Slavian Radev in 1980, Lyubomir Ivanov in 1981 and Solomon Passy in 1985. All three of them had been graduate students at the Sector of Mathematical Logic before, and entered its staff immediately after receiving their Ph.D. degrees (at the Warsaw University in the case of Radev and at the Sofia University in the case of Ivanov and Passy).

The interest of the sector's members in the connections between mathematical logic and theoretical computer science can be seen from the survey [9], a joint work of all the staff of the sector as it was about 1984 together with two Ph.D. students. The bibliography of the survey contains 58 references (without claims on exhaustiveness) to relevant works of foreign authors and 37 references to publications in the mentioned scientific area with results obtained by members of the sector and by other participants in the seminars of the sector.

All people who entered the staff of the sector had high professional qualities, and they did first class research that was acknowledged abroad. In particular, the results from Ivanov's dissertation were published in England in 1986 in his monograph [11]. (These results present a rather interesting algebraic generalization of Recursion Theory, different from the one given in my book [6].) All these people contributed very much to the high level of the scientific and educational activities of the sector. Unfortunately, no other new members of the sector came during its existence, although five other excellent Ph.D. dissertations of our former graduate students were defended in the same period of time, and efforts were made to appoint their authors – Jordan Zashev, Ivan Soskov, Angel Ditchev, Tinko Tinchev and Valentin Goranko. (These dissertations were defended in 1983, 1984, 1984, 1986 and 1988, respectively.) It is difficult to give an explanation of this injustice, except by repeating once more that quite many things did not depend on the will of the sector's chief. A partial correction of the situation was achieved in 1987–1988, when, thanks to the exceptional activity of professor Dimiter Vakarelov, a Laboratory of Applied Logic was created at the Sofia University, and all of the five persons mentioned above became its members. To come to an end of the story about the staff of the sector, I shall list those people who were there in 1989, the last year of existence of the sector. They are (in alphabetic order of surnames): Anatoly Buda, Lyubomir Ivanov, Solomon Passy, Petio Petkov, Dimiter Skordev, Vladimir Sotirov and Dimiter Vakarelov. (Buda, Ivanov, Petkov and Vakarelov were already associate professors at that time, and Skordev was Dr. habil. in mathematics and a full professor.) Three people are missing – Radoslav Pavlov, who left the sector about 1978, and Georgi Gargov and Slavian Radev, who left it about ten years later. (All three of them also were associate professors in 1989.)

The Sector of Mathematical Logic has had a very intensive teaching activity. It has been directed mostly to graduate students. They have been taught the main topics in mathematical logic and recursion theory, as well as many topics of current research by the members of the sector. (A lot of things from mathematical logic taught to these students in the period from 1975 to 1980 are presented in Professor P. Petkov's book [10].) A series of 33 master theses, most of them excellent, were defended during the existence of the sector. The authors of 12 of them have been later (for some time) or still are members of the sector or of some of the two units that descended from it in 1989. If we are to encompass also the period after 1989 too, then we should add 32 more master theses of the same quality with 4 of their authors being members now of some of the two above-mentioned units.

Several conferences were organized by the Sector of Mathematical Logic in the period of its existence. (Three other conferences were organized later by the successors of the sector, namely one in 1990, another one in 1996 and, finally, the present one.) Here follows brief information on the first several conferences.

The very first of them was a Summer School on Algebra and Logic, organized together with the Sector of Algebra of the United Centre of Mathematics. The school took place in September 1979 in Blagoevgrad. The invited lecturers from abroad for the logical part of the school were H. Rasiowa and A. Skowron from Warsaw.

In September 1980, a Conference on Mathematical Logic, dedicated to the memory of A. A. Markov, took place in Sofia. N. M. Nagorny from Moscow presented a talk (prepared jointly with N. A. Shanin) on the works of Markov in mathematical logic and the theory of algorithms. Most of the talks given on the conference were published in [8]. (Unfortunately, no written copy of Nagorny-Shanin's talk was presented for publication in that volume.)

A Summer School on Mathematical Logic and Its Applications took place in September 1983 in Primorsko. Here is the complete list (extracted from [7]) of the participants from abroad (22 persons, including not only the invited lecturers, but also the other participants): B. R. Boričić (Beograd), W. Dańko (Bialystok),

O. Demuth (Prague), Ju. L. Ershov (Novosibirsk), A. Gajda (Bialystok), G. Georgescu (Bucharest), S. S. Goncharov (Novosibirsk), J. Harrera (Paris), A. Jankowski (Warsaw), K. P. Jantke (Berlin), J. Krempa (Warsaw), A. Ju. Muravitsky (Kishinev), N. M. Nagorny (Moscow), S. Puczylowski (Warsaw), H. Rasiowa (Warsaw), C. Rauszer (Warsaw), A. Salwicki (Warsaw), A. Skowron (Warsaw), D. I. Sviridenko (Novosibirsk), A. Szalas (Warsaw), H. Thiele (Berlin), P. Vopěnka (Prague)[3].

A Summer School and Conference on Mathematical Logic honourably dedicated to the 80th anniversary of Kurt Gödel took place in Druzhba near Varna from September 24 to October 4, 1986. The list of invited lecturers includes 20 persons, namely: C. C. Christian, J. W. Dawson, Jr., P. P. Petkov, J. van Benthem, D. S. Bridges, O. Demuth, A. G. Dragalin, Yu. L. Ershov, S. S. Goncharov, H. R. Jervell, Y. N. Moschovakis, N. M. Nagorny, V. A. Nepomnyashchy, N. A. Shanin, A. Skowron, H. Rasiowa, G. Sambin, K. Segerberg, B. A. Trakhtenbrot, V. A. Uspensky (there was also an invited seminar talk by D. Siefkes). The corresponding proceedings [12] include 14 of the invited papers and 13 of the contributed ones. The event was estimated by many of the participants as very successful, and a person to be certainly thanked for the success was Professor Petio Petkov whose activity at organizing the event was highly productive.

The last event of this kind during the existence of the sector was a Summer School and Conference on Mathematical Logic honourably dedicated to the 90th anniversary of Arend Heyting. It took place in September 1988, again near Varna. Petio Petkov was the chairman of the Organizing Committee, and the event was very successful too. A proceedings volume [14] was published again. It contains two invited papers on intuitionism and Heyting, 12 invited lectures and 14 selected contributed papers. The invited papers and lectures included in the volume are by the following people: A. S. Troelstra, D. van Dalen, D. de Jongh (joint with F. Veltman), S. Hayashi, B. Kushner, G. Mints, D. Normann, H. Ono, V. Shehtman (joint with D. Skvortsov), I. N. Soskov, G. Takeuti, W. Veldman, A. Visser, S. S. Wainer.

Along with the conferences, the sector also had an activity of another nature, namely a series of nine popular lectures in 1987, followed by a discussion. The lectures and the discussion attracted a very large audience and surely helped some more students to make their choice in favour of the mathematical logic. The realization of this activity resulted in the book [13].

In July 1989, after long previous discussions in other places, the Faculty Council of the Faculty of Mathematics took the decision a Department of Mathematical Logic and Its Applications to be formed at the faculty. It was to include the people from the Sector of Mathematical Logic who administratively belonged to the faculty, i.e. Petkov, Skordev and Vakarelov, as well as Buda, who moved from the Institute of Mathematics into the Faculty of Mathematics at that time, the five members of the Laboratory of Applied Logic, namely Ditchev, Goranko, Soskov, Tinchev, Zashev and, in addition, Roussanka Loukanova, who came into the new department

[3] It seems, however, that P. Vopěnka in fact did not attend the event.

from the former Sector of Mathematical Linguistics. Skordev was chosen to be the chief again. The other people from the Sector of Mathematical Logic, i.e. Lyubomir Ivanov, Solomon Passy and Vladimir Sotirov, formed a Section of Mathematical Logic at the Institute of Mathematics with Lyubomir Ivanov as its chief.

The staff of both units underwent some changes in the next years. For example, Marion Mircheva was a member of the Section of Logic during a certain period of time, Solomon Passy left the section at a certain moment and devoted himself completely to politics, Dimiter Dobrev and Dimitar Guelev came into the section later. (Guelev did this after defending his Ph.D. dissertation at the Sofia University.) Several new persons came into the department at the faculty: Stela Nikolova in 1991, Alexandra Soskova in 1994, Vessela Baleva in 1998, Anton Zinoviev in 1999. Three of them had to replace Valentin Goranko, Jordan Zashev and Roussanka Loukanova, who had left the department at different times within this period. (Zashev moved to the Section of Logic, whereas Goranko and Loukanova got academic positions abroad.) Meanwhile Soskova, Nikolova and recently Baleva defended their Ph.D. dissertations at the Sofia University, and Loukanova defended one at the Moscow State University when she still was a member of the department. Soskov, Ditchev and Tinchev (as well as Zashev and Sotirov in the Section of Logic) became associate professors. Vakarelov and Soskov received also the Dr. habil. degree in this period, and Vakarelov became a full professor. In connection with the age restrictions imposed by the law the chief position in the department was eventually taken by Ivan Soskov. Unlike before 1989, now the department has many teaching activities in the first stage years of education (the Bachelor Program). The teaching traditions from the past have been continued by the department's activities in the M.Sc. Program.

At the end, I would like to say a few concluding words. To my opinion, history must be known and respected. Unfortunately, the department's archives turned out to be not in a sufficiently good state. Of course, this is mainly my fault. I am sorry for all imperfections and possible incorrectnesses that resulted from this in the present notes. I recommend to my younger colleagues to take care about preserving and saving all essential information about what happens in the department in order that this information could be used by those, who will come later.

Acknowledgements. I thank professor Petio Petkov for his very substantial help in the search of sources throwing light on the pre-history and the early history of development of mathematical logic in Sofia. Thanks are due also to him and to professor Dimiter Vakarelov for clarifying when a lecture course in mathematical logic has been first given in Sofia, and to professor Vladimir Sotirov for giving detailed information about the staff of the Section of Mathematical Logic as in 1970. I would like also to mention that Vakarelov's paper [15] has been very useful in preparing these notes, and the English version of their presentation has got numerous serious improvements thanks to Dr. Dimitar Guelev's competent and precise editorial work on a draft of the notes.

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
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SOME RESULTS ON THE UPPER-SEMILATTICE OF R.E. m-DEGREES INSIDE A SINGLE R.E. tt-DEGREE

ANGEL V. DITCHEV

In this paper it is shown that: 1) There exists such a tt-degree which contains infinitely many m-degrees of the type of ω^k for any positive natural number k; 2) There exists such a tt-degree which contains infinitely many m-degrees of the type of $\mathbb{Q}$ of rational numbers.

Keywords: m-degrees, tt-degrees, upper semilattice

2000 MSC: 03D25, 03D30, 03D45, 03C57

In [1 – 7] Degtev, Ditchev and Downey have considered how many recursively enumerable (r.e.) m-degrees could be contained in a single r.e. tt-degree. It is shown in [2, 5, 6] that a single r.e. tt-degree can contain finitely many m-degrees, and in [3, 4, 7] — infinitely many r.e. m-degrees. In the case when a single r.e. tt-degree contains infinitely many r.e. m-degrees, it is known that they can be linearly ordered in the type of the ordinal ω [7] and can be mutually incomparable [3, 4]. In the present paper we show that a single r.e. tt-degree (even pc-degree) can contain infinitely many m-degrees with type the ordinal ω^k for every natural number k and with type $\mathbb{Q}$ of the rational numbers.

In this paper we use $\mathbb{N}$ to denote the set of all natural numbers, $\mathbb{Z}$ — the set of all integers, and $\mathbb{Q}$ — the set of all rational numbers. We use also ω^k to denote the usual ordinal number.

If f is a partial function, we use $\mathrm{Dom}(f)$ to denote the domain and $\mathrm{Ran}(f)$ — the range of values of the function f.

If A is a finite set, we use $|A|$ to denote the cardinality of the set A.

Let us remind some definitions from [9, 10] and give some new ones.

Let Π, L, R be the usual primitive recursive functions such that $\mathrm{Dom}(\Pi) = \mathbb{N}^2$, $\mathrm{Ran}(\Pi) = \mathrm{Dom}(\mathrm{L}) = \mathrm{Dom}(\mathrm{R}) = \mathrm{Ran}(\mathrm{L}) = \mathrm{Ran}(\mathrm{R}) = \mathbb{N}$, which satisfy the following equations for all natural numbers x, y:

$$\mathrm{L}(\Pi(x,y)) = x, \ \mathrm{R}(\Pi(x,y)) = y, \ \Pi(\mathrm{L}(x), \mathrm{R}(x)) = x.$$

If β is a Goedel function, then for every natural numbers k, $p_1, \ldots, p_k$, i, $k > 0$, we use the following notations:

$$\langle p_1, \ldots, p_k \rangle = \mu p[\beta(p,0) = k \& \beta(p,1) = p_1 \& \ldots \& \beta(p,k) = p_k];$$

$$\mathrm{lh}(p) = \beta(p,0); \quad (p)_i = \beta(p, i+1);$$

$$\mathrm{Seq}(p) \iff \forall x(x < p \Rightarrow (\mathrm{lh}(x) \neq \mathrm{lh}(p) \vee \exists i(i < \mathrm{lh}(p) \& (x)_i \neq (p)_i)));$$

$$\mathrm{Seq}_k(p) \iff \mathrm{Seq}(p) \& \mathrm{lh}(p) = k.$$

$\langle p_1, \ldots, p_k \rangle$ is a code of the sequence $p_1, \ldots, p_k$, $\mathrm{lh}(p)$ — the length of the sequence with code p, and Seq and Seq_k are predicates, which indicate a sequence and a sequence with length k, respectively.

A set A is said to be m-reducible to a set B $(A \leq_m B)$ iff there exists a total recursive function f such that the following equivalence hold:

$$\forall x(x \in A \iff f(x) \in B).$$

The set A is said to be *bounded conjunctive reducible* (bc-reducible) to the set B iff there exist natural number k and k total recursive functions $f_1, \ldots, f_k$, which satisfy the following equivalence:

$$\forall x[x \in A \iff f_1(x) \in B \& \ldots \& f_k(x) \in B].$$

If r is any reducibility, a set A is said to be *r-equivalent* to a set B $(A \equiv_r B)$ iff $A \leq_r B$ and $B \leq_r A$. For any reducibility r the *r-degree* of the set A is called the family $d_r(A) = \{B | B \equiv_r A\}$. If some *$r$-degree* contains a set A, which is recursively enumerable, then this *r-degree* is said to be recursively enumerable (r.e.).

The ordinal ω^k we represent as the set $\{(a_1, \ldots, a_k) | a_1 \in \mathbb{N} \& \ldots \& a_k \in \mathbb{N}\}$ and the order is the usual lexical one:

$$(a_1, \ldots, a_k) \prec (b_1, \ldots, b_k) \iff$$

$$(a_1 < b_1) \vee (a_1 = b_1 \& a_2 < b_2) \vee \ldots \vee (a_1 = b_1 \& a_2 = b_2 \& \ldots \& a_{k-1} = b_{k-1} \& a_k < b_k).$$

We are constructing an r.e. bc-degree, which considered as an upper-semilattice of m-degree contains a set of type ω^k of different r.e. m-degrees. The idea for constructing such r.e. bc-degree comes from the effective structures with functions and without predicates. The functions are choosen in an appropriate way to ensure that the choosen sets are in the same bc-degree and in the above-mentioned order.

For the sake of simplicity, we consider in full only the case $k = 2$.

Let $\{\theta_k\}$, $k \in \mathbb{N}$, be the recursive functions with $\mathrm{Dom}(\theta_k) = \mathrm{Ran}(\theta_k) = \omega^2$, $k \in \mathbb{N}$, defined as follows:

$$\theta_0(i,j) = (i, j+1), \quad i,j \in \mathbb{N};$$

$$\theta_1(i,j) = \begin{cases} (0,j), & \text{if } i = 0, \\ (i-1,0), & \text{if } i > 0 \ \& \ j \text{ is even}, \\ (i-1,2), & \text{if } i > 0 \ \& \ j \text{ is odd }; \end{cases}$$

$$\theta_2(i,j) = \begin{cases} (0,1), & \text{if } i = 0 \ \& \ j = 0, \\ (0,j), & \text{if } i = 0 \ \& \ j > 0, \\ (i-1,0), & \text{if } i > 0 \ \& \ (j \text{ is odd} \vee j = 0), \\ (i-1,1), & \text{if } i > 0 \ \& \ j \text{ is even } \ \& \ j > 0; \end{cases}$$

$$\theta_{k+3}(i,j) = \begin{cases} (i+1,0), & \text{if } j = k, \\ (i+1,k), & \text{if } j = 0, \\ (i+1,j), & \text{if } j \notin \{0,k\}, \end{cases}$$

$k \in \mathbb{N}$.

It is easy to check that the following lemmas are correct.

Lemma 1. *For all $a \in \omega^2$ and for all natural numbers i, j and k the following equivalences hold:*

$$a = (i,j) \iff \theta_0(a) = (i, j+1);$$

$$a = (i,k) \iff \theta_{k+3}(a) = (i+1, 0);$$

$$a = (i+1,0) \iff \theta_1(a) = (i,0) \ \& \ \theta_2(a) = (i,0).$$

Lemma 2. *For all $a, b \in \omega^2$, such that $a \preceq b$, there exists a function η, which is a composition of the functions θ_0, $\{\theta_{k+3}\}_{k\in\mathbb{N}}$,* id *such that $\forall c (c = a \iff \eta(c) = b)$.*

Lemma 3. a) *For all natural numbers i, j such that $i < j$ there exist functions $\eta_1, \ldots, \eta_{2l}$, which are compositions of the functions θ_1, θ_2, such that $\forall a(a = (j,0) \iff \eta_1(a) = (i,0) \ \& \ldots \& \ \eta_{2l}(a) = (i,0)$.*

b) *For all $a, b \in \omega^2$ such that $a \prec b$ there exist functions $\eta_1, \ldots, \eta_{2l}$, which are compositions of the functions $\{\theta_k\}_{k\in\mathbb{N}}$, such that $\forall c(c = b \iff \eta_1(c) = a \ \& \ldots, \& \ \eta_{2l}(c) = a)$.*

We say that a set A contains almost all even (odd) numbers iff there exists a finite set B such that all even (odd) numbers are subset of $A \cup B$.

Lemma 4. a)*If $\theta_k(i+1,j) = (i_1, j_1)$, $k = 1, 2$, for some natural numbers i_1, j_1, then either for almost all even numbers j' or for almost all odd numbers j' the equation $\theta_k(i+1, j') = (i_1, j_1)$ holds.*

b) *Let η be such composition of the functions $\{\theta_k\}_{k\in\mathbb{N}}$ that at least one of θ_1 and θ_2 appears in η. If $\eta(i+1,j) = (i_1, j_1)$ for some natural numbers i_1, j_1, then either for almost all even numbers j' or for almost all odd numbers j' the equation $\eta(i+1,j') = (i_1, j_1)$ holds.*

Let $\varphi_i = \langle i, x\rangle$, $i \in \mathbb{N}$, and $N_0 = \mathbb{N} \setminus (\cup_{i\in\mathbb{N}} \operatorname{Ran}(\varphi_i))$.

Definition. Let $\{A_a\}_{a\in\omega^2}$ be a sequence of disjoint subsets of N_0. We define the sequence $\{[A_a]\}_{a\in\omega^2}$ of disjoint sets of natural numbers by the following rules:
(a) If $p \in A_a$, then $p \in [A_a]$;
(b) If $i \in \mathbb{N}$, $p \in [A_a]$, and $\theta_i(a) = b$, then $\varphi_i(p) \in [A_b]$.

Lemma 5. *If $\{A_a\}_{a\in\omega^2}$ is a recursive (r.e.) sequence of disjoint subsets of N_0, then $\{[A_a]\}_{a\in\omega^2}$ is a recursive (r.e.) sequence of disjoint sets.*

Lemma 6. *If $\{A_a\}_{a\in\omega^2}$ is a sequence of disjoint subsets of N_0, then the following equivalences hold for all natural x, i, j:*

$$x \in [A_{(i,j)}] \iff \varphi_0(x) \in [A_{(i,j+1)}];$$

$$x \in [A_{(i,k)}] \iff \varphi_{k+3}(x) \in [A_{(i+1,0)}];$$

$$x \in [A_{(i+1,0)}] \iff \varphi_1(x) \in [A_{(i,0)}] \;\&\; \varphi_2(x) \in [A_{(i,0)}].$$

Corollary 1. *If $\{A_a\}_{a\in\omega^2}$ is a sequence of disjoint subsets of N_0, then $[A_{(i,j)}] \leq_m [A_{(i,j+1)}]$ and $[A_{(i,k)}] \leq_m [A_{(i+1,0)}]$ for all natural numbers i, j.*

Corollary 2. *If $\{A_a\}_{a\in\omega^2}$ is a sequence of disjoint subsets of N_0, then $[A_a] \equiv_{bc} [A_b]$ for all $a, b \in \omega^2$.*

Corollary 3. *If $\{A_a\}_{a\in\omega^2}$ is a sequence of disjoint subsets of N_0, then $[A_a] \equiv_{tt} [A_b]$ for all $a, b \in \omega^2$.*

Lemma 7. *For every natural number x, either $x \in N_0$ or there exists an effective way to find a function φ, which is a composition of the functions $\{\theta_k\}_{k\in\mathbb{N}}$ and $y \in N_0$ such that $\varphi(y) = x$.*

Lemma 8. *Let $\{A_a\}_{a\in\omega^2}$ be a sequence of disjoint subsets of N_0. For any function φ, which is a composition of the functions $\{\theta_k\}_{k\in\mathbb{N}}$, and for any $a \in \omega^2$ there exists $b \in \omega^2$ such that $\varphi([A_a]) \subseteq [A_b]$.*

Lemma 9. *Let $\{A_a\}_{a\in\omega^2}$ be a sequence of disjoint non-empty subsets of N_0. For any function φ, which is a composition of the functions $\{\theta_k\}_{k\in\mathbb{N}}$, and for any $a, b \in \omega^2$ there exists an effective way to verify whether or not $\varphi([A_a]) \subseteq [A_b]$.*

Lemma 10. *Let $\{A_a\}_{a\in\omega^2}$ be a sequence of disjoint non-empty subsets of N_0 and φ be a composition of the functions $\{\theta_k\}_{k\in\mathbb{N}}$. If $a, b \in \omega^2$ are such that $\varphi([A_a]) \subseteq [A_b]$, then there exist infinitely many $c \in \omega^2$ such that $\varphi([A_c]) \subseteq [A_b]$.*

Let $N_0 = N_1 \cup N_2$, where N_1 and N_2 are infinite disjoint recursive sets, and let r' be a monotonically increasing function such that $\mathrm{Ran}(r') = N_1$ and $r(n) = r'(\frac{n(n+1)}{2} + n)$. In addition, let Φ be a partial recursive function (p.r.f.), which is universal for all unary p.r.f. Let $\Phi_e = \lambda x.\Phi(e, x)$ and $\Phi_{e,s}$ be a finite p.r. approximation of Φ_e, i.e.

$$\Phi_{e,s}(x) \cong \begin{cases} \Phi_e(x), & \text{if } x \in \mathrm{Dom}(\Phi_e) \& \ \Phi_e(x) \text{ is computable in less than } s \text{ steps,} \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Theorem 1. *There exists an r.e. bc-degree, which contains different m-degrees of the type of ω^2.*

Proof. In order to construct such a degree, we shall construct an r.e. sequence $\{A_a\}_{a\in\omega^2}$ of disjoint subset of N_0 such that if $a \prec b$, then $[A_a] \leq_m [A_b]$, but $[A_b] \not\leq_m [A_a]$. Then it will follow from Corollary 2 that all sets $\{A_a\}_{a\in\omega^2}$ are in the same *bc*-degree and, therefore, the proof will be completed.

We construct the sets $\{A_a\}_{a\in\omega^2}$ by steps, building a finite approximation $A_{a,s}$ of A_a on step s, $a \in \omega^2$.

On step s, if $(s)_0 = \langle e, i, j, i_1, j_1\rangle$ and $(i, j) \prec (i_1, j_1)$, then our aim is to satisfy that the function Φ_e does not m-reduce $[A_{(i_1,j_1)}]$ to $[A_{(i,j)}]$, i.e. to find such a witness $x \in \mathrm{Dom}(\Phi_e)$ that at least one of the following two conditions is satisfied:

(i) $x \notin [A_{(i_1,j_1)}]$ & $\Phi_e(x) \in [A_{(i,j)}]$;

(ii) $x \in [A_{(i_1,j_1)}]$ & $\Phi_e(x) \notin [A_{(i,j)}]$.

For this purpose on step s if we find an x, such that $x \in \mathrm{Dom}(\Phi_e)$, then we would like to do the following:

If $\Phi_e(x) \in [A_{(i,j)}]$, then to put x outside of $A_{(i_1,j_1)}$, satisfying (i).

If $\Phi_e(x) \notin [A_{(i,j)}]$, then to put x in $A_{(i_1,j_1)}$, satisfying (ii).

If on step s, x is placed in some set A_a in order to satisfy either (i) or (ii), we create an $(s)_0$-*requirement* x. In this case, if x satisfies (ii), we need also some element y not to belong to a chosen set A_a. So, we create a *negative* $(s)_0$-*requirement* y. To guarantee that for any e, such that Φ_e is a total, and for every $(i, j), (i_1, j_1)$, such that $(i, j) \prec (i_1, j_1)$, there exists an x satisfying either (i) or (ii), we shall use the priority argument, so that the smaller $(s)_0$ will have priority.

If x is an $(s)_0$-requirement and y is a negative $(s)_0$-requirement, created on step s, and till step t the condition (ii), which is satisfied on step s, is not injured, then we say that the $(s)_0$-requirement and the negative $(s)_0$-requirement are *active* on step t.

If an $(s)_0$-requirement x satisfies (i), then we call it *active* on any step $t > s$.

If an $(s)_0$-requirement (a negative $(s)_0$-requirement) created on step s is active on every step $t > s$, then we say that it is a *constant*.

Now we can describe the construction of the sequence $\{A_a\}_{a\in\omega^2}$.

Step $s = 0$. Let $N_2 = \{a_0 < a_1 < \ldots\}$; we take $A_{(i,j),0} = \{a_{\Pi(i,j)}\}$. Thus it is ensured that $A_{(i,j)}$ is non-empty.

Step $s > 0$. If neither $\mathrm{Seq}((s)_0)$ nor $\mathrm{Seq}_5((s)_0) \,\&\, (((s)_0)_1, ((s)_0)_2) \not\prec (((s)_0)_3, ((s)_0)_4)$, then we do nothing, i.e. we take $A_{(i,j),s} = A_{(i,j),s-1}$, $i, j \in \mathbb{N}$, and do not create any requirements.

If $\mathrm{Seq}_5((s)_0)$ and $s = \langle e, i, j, i_1, j_1\rangle$, where $(i, j) \prec (i_1, j_1)$, we verify whether an active $(s)_0$-requirement exists. If there exists such a requirement, then do nothing.

If such a requirement does not exist, then we verify whether there exists an $x \in N_1$ such that $x > r((s)_0)$, $x \in \mathrm{Dom}(\Phi_{e,s})$, $x \notin \cup_{a\in\omega^2} A_{a,s-1}$ and x does not belong to any active negative requirement, created on a step $t < s$ such that $(t)_0 < (s)_0$. If such an x does not exist, then we do nothing.

Otherwise, we denote by x_s the least such x and create an $(s)_0$-requirement x_s. Let $\Phi_e(x_s) = z$ and $\psi(y) = z$, where ψ is either a composition of the functions $\{\varphi_k\}_{k\in\mathbb{N}}$ or $\psi = \mathrm{id}$ and $y \in N_0$.

We verify whether $z \in A_{(i,j),s-1}$. If so, then we fix $A_{(i,j),s} = A_{(i,j),s-1} \cup \{x_s\}$, $A_{(k,l),s} = A_{(k,l),s-1}$ for $(k, l) \neq (i, j)$.

Otherwise, we verify if $z \in A_{(i',j'),s-1}$ for some $(i', j') \neq (i, j)$. If so, then fix $A_{(i_1,j_1),s} = A_{(i_1,j_1),s-1} \cup \{x_s\}$, $A_{(k,l),s} = A_{(k,l),s-1}$ for $(k, l) \neq (i_1, j_1)$. Otherwise we consider two cases:

Case I. $x_s \neq y$. We fix $A_{(i_1,j_1),s} = A_{(i_1,j_1),s-1} \cup \{x_s\}$, $A_{(k,l),s} = A_{(k,l),s-1}$ for $(k, l) \neq (i_1, j_1)$ and create a negative $(s)_0$-requirement y.

Case II. $x_s = y$. We find effectively $(i_2, j_2) \neq (i_1, j_1)$ such that $\psi([A_{(i_2,j_2)}]) \subseteq [A_{(i,j)}]$ and fix $A_{(i_2,j_2),s} = A_{(i_2,j_2),s-1} \cup \{x_s\}$, $A_{(k,l),s} = A_{(k,l),s-1}$ for $(k, l) \neq (i_2, j_2)$.

Finally, we take $A_a = \cup_{s\in\mathbb{N}} A_{a,s}$, $a \in \omega^2$.

Obviously, the construction is effective, hence the sequence $\{A_a\}_{a\in\omega^2}$ is r.e. Moreover, $\{A_a\}_{a\in\omega^2}$ is a sequence of disjoint subsets of N_0 since one element may be placed in only one A_a.

Lemma 11. *The set* $N_1 \setminus A$ *is infinite.*

Proof. Let $(N_1)_n = \{x | x \in N_1 \& x < n\}$.

We will prove that the set $(N_1)_{r(n)} \cap (N_1 \setminus A)$ contains at least n elements or, which is the same, $|(N_1)_{r(n)} \cap A| \leq \dfrac{n(n+1)}{2}$.

Indeed, for every $\langle e, i, j, i_1, j_1\rangle$ such that $(i, j) \prec (i_1, j_1)$ we have no more than $\langle e, i, j, i_1, j_1\rangle + 1$ $\langle e, i, j, i_1, j_1\rangle$-requirements and each of them is greater than $r(\langle e, i, j, i_1, j_1\rangle)$ and belongs to some $A_a \subseteq A$. Therefore, in $|(N_1)_{r(n)} \cap A|$ there are only m-requirements for $m < n$, i.e. in $|(N_1)_{r(n)} \cap A|$ there are no more than $1 + 2 + \ldots + n = \dfrac{n(n+1)}{2}$ elements. Lemma 11 is proved.

Lemma 12. *The set $N_1 \setminus A$ is immune, i.e. $N_1 \setminus A$ does not contain infinite r.e. subset.*

Proof. Let us assume that there exists a set $C \subseteq N_1 \setminus A$, which is infinite and r.e. and $x_0 \in N_2$. Obviously,

$$f(x) = \begin{cases} x_0, & \text{if } x \in C, \\ \text{undefined}, & \text{otherwise} \end{cases}$$

is a p.r.f. Let e be a natural number, such that $f = \Phi_e$ and let $x \in \mathrm{Dom}(f)$ be such that $x > r(\langle e, 0, 1, 0, 2\rangle)$ and s_0 be the least s satisfying the equality $\Phi_{e,s}(x) = f(x)$. Then x must be an $\langle e, 0, 1, 0, 2\rangle$-requirement created on some step $s > s_0$ such that $(s)_0 = \langle e, 0, 1, 0, 2\rangle$, i.e. $C \cap A$ is non-empty, which contradicts the assumption. Therefore, $N_1 \setminus A$ is immune.

Lemma 13. *For any natural number e, such that $N_1 \subseteq \mathrm{Dom}(\Phi_e)$, and for every $\langle e, i, j, i_1, j_1\rangle$, such that $(i, j) \prec (i_1, j_1)$, there exists a constant $\langle e, i, j, i_1, j_1\rangle$-requirement.*

Proof. Assume that there is no constant $\langle e, i, j, i_1, j_1\rangle$-requirement, where $(i, j) \prec (i_1, j_1)$ and $N_1 \subseteq \mathrm{Dom}(\Phi_e)$. We find an s_0 such that if $s \geq s_0$ and $\langle e', i', j', i'_1, j'_1\rangle < \langle e, i, j, i_1, j_1\rangle$, then every constant $\langle e', i', j', i'_1, j'_1\rangle$-requirement is already created. Moreover, let $x \in N_1 \setminus A$, $x > r(\langle e, i, j, i_1, j_1\rangle)$ and s be such that $s \geq s_0$, $\Phi_{e,s}(x)\Phi_e(x)$ and $(s)_0 = \langle e, i, j, i_1, j_1\rangle$. Then on step s a constant $\langle e, i, j, i_1, j_1\rangle$-requirement x would be created. Lemma 13 is proved.

Now we will prove Theorem 1. Let us assume that $[A_{(i_1,j_1)}] \leq_m [A_{(i,j)}]$ and $(i, j) \prec (i_1, j_1)$. Therefore, there exists a total recursive function f such that $\forall x(x \in [A_{(i_1,j_1)}] \iff f(x) \in [A_{(i,j)}])$.

Let e be such that $\Phi_e = f$. It follows from Lemma 13 that there exists a constant $\langle e, i, j, i_1, j_1\rangle$-requirement x_s, created on step s. Then $x_s \in N_1$, $f(x_s) = z$, $z = \psi(y)$, where ψ is either a composition of $\{\varphi_k\}_{k \in \mathbb{N}}$ or $\psi = \mathrm{id}$, and $y \in N_0$.

It is not possible $z \in A_{(i,j),s-1}$, because then $x_s \in A_{(i,j),s}$, since $x_s \in A_{(i,j)} \subseteq [A_{(i,j)}]$ and $f(x_s) = z \in [A_{(i,j)}]$.

It is also not possible $z \in A_{(i',j'),s-1}$ for some $(i', j') \neq (i, j)$, because then $x_s \in A_{(i_1,j_1),s} \subseteq A_{(i_1,j_1)} \subseteq [A_{(i_1,j_1)}]$ and $f(x_s) = z \in [A_{(i_1,j_1)}]$.

It is also not possible $x_s \neq y$, because $x_s \in A_{(i_1,j_1),s} \subseteq A_{(i_1,j_1)} \subseteq [A_{(i_1,j_1)}]$ and $f(x_s) = z \notin \cup_{a \in \omega^2}[A_a]$.

Therefore, $x_s = y$. Then $(i_2, j_2) \neq (i_1, j_1)$, $\psi([A_{(i_2,j_2)}]) \subseteq [A_{(i,j)}]$ and $x_s \in A_{(i_2,j_2),s} \subseteq A_{(i_2,j_2)} \subseteq [A_{(i_2,j_2)}]$. The received contradiction shows that the assumption $[A_{(i_1,j_1)}] \leq_m [A_{(i,j)}]$ is not true. Theorem 1 is proved.

Corollary 4. *There exists an r.e. tt-degree, which contains different m-degrees of the type of ω^2.*

Now we consider the corresponding functions for the case $k \in \mathbb{N}, k > 2$ — $\{\theta^n_m\}$, θ_0, θ_1, θ_2 with $\mathrm{Dom}(\theta^n_m) = \mathrm{Ran}(\theta^n_m) = \mathrm{Dom}(\theta_l) = \mathrm{Ran}(\theta_l) = \omega^k$, $m \in \mathbb{N}$,

$n = 1, \ldots, k-1$, $l = 0, 1, 2$ defined as follows:

$$\theta_0(i_1, \ldots, i_k) = (i_1, \ldots, i_{k-1}, i_k + 1), \quad i_1, \ldots, i_k \in \mathbb{N};$$

$$\theta_1(i_1, \ldots, i_k) = \begin{cases} (0, i_2, \ldots, i_k), & \text{if } i_1 = 0, \\ (i_1 - 1, 0, \ldots, 0), & \text{if } i_1 > 0 \ \& \ i_2 \text{ is even}, \\ (i_1 - 1, 2, 0, \ldots, 0), & \text{if } i_1 > 0 \ \& \ i_2 \text{ is odd}; \end{cases}$$

$$\theta_2(i_1, \ldots, i_k) = \begin{cases} (0, 1, i_3, \ldots, i_k), & \text{if } i_1 = 0 \ \& \ i_2 = 0, \\ (0, i_2, \ldots, i_k), & \text{if } i_1 = 0 \ \& \ i_2 > 0, \\ (i_1 - 1, 0, \ldots, i_k), & \text{if } i_k > 0 \ \& \ (i_2 \text{ is odd } \vee i_2 = 0), \\ (i_1 - 1, 1, 0, \ldots, 0), & \text{if } i_1 > 0 \ \& \ i_2 \text{ is even } \ \& \ i_2 > 0; \end{cases}$$

$$\theta_l^1(i_1, \ldots, i_k) = \begin{cases} (i_1, \ldots, i_{k-1} + 1, 0), & \text{if } i_k = l, \\ (i_1, \ldots, i_{k-1}, l), & \text{if } i_k = 0, \\ (i_1, \ldots, i_{k-1} + 1, i_k), & \text{if } i_k \notin \{0, l\}, \end{cases}$$

$$\theta_l^{k-1}(i_1, \ldots, i_k) = \begin{cases} (i_1 + 1, 0, i_3, \ldots, i_k), & \text{if } i_2 = l, \\ (i_1 + 1, l, i_3, \ldots, i_k), & \text{if } i_2 = 0, \\ (i_1 + 1, i_2, \ldots, i_k), & \text{if } i_2 \notin \{0, l\}, \end{cases}$$

$l \in \mathbb{N}$.

Analogously, one can prove the following

Theorem 2. *There exists an r.e. bc-degree, which contains different m-degrees of the type of* ω^k *for any positive integer* k.

Corollary 5. *There exists an r.e. tt-degree, which contains different m-degrees of the type of* ω^k *for any positive integer* k.

We will construct also an r.e. *bc*-degree, which considered as an upper-semi-lattice of *m*-degree contains a set of type $\mathbb{Q}$ of different r.e. *m*-degree. The idea for constructing such r.e. *bc*-degree is the same as in Theorem 1.

Let $\mathbb{Q}$ be the set of rational numbers with the usual ordering, Q be the set $\{(a_1, a_2 + 1) | a_1 \in \mathbb{Z} \& a_2 \in \mathbb{N}\}$. It is well-known that we can represent $\mathbb{Q}$ with the elements of Q having in mind that two elements $(a_1, a_2), (b_1, b_2)$ represent the same rational number $(\frac{a_1}{a_2 + 1})$ iff $a_1.(b_2 + 1) = (a_2 + 1).b_1$. We write

$$(a_1, a_2) \prec (b_1, b_2) \text{ iff } \frac{a_1}{a_2 + 1} < \frac{b_1}{b_2 + 1} \quad \text{and write}$$

$$(a_1, a_2) \preceq (b_1, b_2) \text{ iff } (\frac{a_1}{a_2 + 1} < \frac{b_1}{b_2 + 1} \text{ or } \frac{a_1}{a_2 + 1} = \frac{b_1}{b_2 + 1}).$$

Let θ_0, θ_1 and θ_2 be the recursive functions with $\mathrm{Dom}(\theta_k) = \mathrm{Ran}(\theta_k) = Q$, $k = 0, 1, 2$, defined as follows:

$$\theta_0(i, k) = (i + 1, k), \ i \in \mathbb{Z}, \ k \in \mathbb{N};$$

$$\theta_1(i, k) = \begin{cases} (i - 3, k), & \text{if } \mathrm{rem}(3, i) = 0, \\ (i - 2, k), & \text{if } \mathrm{rem}(3, i) = 2, \\ (i, k), & \text{if } \mathrm{rem}(3, i) = 1; \end{cases}$$

$$\theta_2(i,k) = \begin{cases} (i-3,k), & \text{if } \operatorname{rem}(3,i)=0, \\ (i-1,k), & \text{if } \operatorname{rem}(3,i)=1, \\ (i,k), & \text{if } \operatorname{rem}(3,i)=2; \end{cases}$$

$i \in \mathbb{Z},\ k \in \mathbb{N}$.

The following lemmas are analogous to those before Theorem 1 and it is easy to check that they are again correct.

Lemma 14. *For all $a \in \mathbb{Q}$, $i \in \mathbb{Z}$ and for every natural number k the following equivalences hold:*

$$a = (i,k) \iff \theta_0(a) = (i,k+1);$$

$$a = (i+3,k) \iff \theta_1(a) = (i,k)\ \&\ \theta_2(a) = (i,k).$$

Lemma 15. *For all $a, b \in Q$ such that $a \preceq b$, there exists a function η, which is a composition of the functions θ_0, θ_1, θ_2,* id *such that* $\forall c(c = a \iff \eta(c) = b)$.

Lemma 16. *For all $a, b \in Q$ such that $a \preceq b$, there exist functions $\eta_1, \ldots, \eta_{2l}$, which are compositions of the functions θ_1, θ_2 such that $\forall c(c = b \iff \eta_1(c) = a\ \&\ldots\&\ \eta_{2l}(c) = a$.*

Lemma 17.a) *If $\theta_k(i_1,j_1) = (i,j)$, $k \in \{1,2\}$ for some integers i, j such that* $\operatorname{rem}(3,i) = 0$, *then $j_1 = j$ and there exists at least one $i_2 \neq i_1$ such that the equation $\theta_k(i_2,j) = (i,j)$ holds.*

b) *Let η be such composition of the functions θ_0 θ_1, θ_2 that at least one of θ_1 and θ_2 appears in η. If $\eta(i_1,j_1) = (i,j)$ for some integers i, j, such that* $\operatorname{rem}(3,i) = 0$, *then $j_1 = j$ and there exists at least one $i_2 \neq i_1$ such that the equation $\eta(i_2,j) = (i,j)$ holds.*

Let $\varphi_i = \langle i, x\rangle$, $i = 0,1,2$; $x \in \mathbb{N}$ and $N_0 = \mathbb{N}\setminus(\operatorname{Ran}(\varphi_0))\cup\operatorname{Ran}(\varphi_1)\cup\operatorname{Ran}(\varphi_2))$.

Definition. Let $\{A_a\}_{a\in Q}$ be a sequence of disjoint subset of N_0. We define the sequence $\{[A_a]\}_{a\in Q}$ of disjoint sets of natural numbers by the following rules:

(a) If $p \in A_a$, then $p \in [A_a]$;

(b) If $i \in \{0,1,2\}$, $p \in [A_a]$ and $\theta_i(a) = b$, then $\varphi_i(p) \in [A_b]$.

Lemma 18. *If $\{A_a\}_{a\in Q}$ is a recursive (r.e.) sequence of disjoint subsets of N_0, then $\{[A_a]\}_{a\in Q}$ is a recursive (r.e.) sequence of disjoint sets.*

Lemma 19. *If $\{A_a\}_{a\in Q}$ is a sequence of disjoint subsets of N_0, then the following equivalences hold for all natural x, j and integer i:*

$$x \in [A_{(i,j)}] \iff \varphi_0(x) \in [A_{(i+1,j)}];$$

$$x \in [A_{(i+3,j)}] \iff \varphi_1(x) \in [A_{(i,j)}]\ \&\ \varphi_2(x) \in [A_{(i,j)}].$$

Corollary 6. *If $\{A_a\}_{a\in Q}$ is a sequence of disjoint subsets of N_0, then $[A_{(i,j)}] \leq_m [A_{(i+1,j)}]$ for all natural j and integer i.*

Corollary 7. *If $\{A_a\}_{a\in Q}$ is a sequence of disjoint subsets of N_0, then $[A_a] \equiv_{bc} [A_b]$ for all $a, b \in Q$.*

Corollary 8. *If $\{A_a\}_{a\in Q}$ is a sequence of disjoint subsets of N_0, then $[A_a] \equiv_{tt} [A_b]$ for all $a, b \in Q$.*

Lemma 20. *For every natural number x, either $x \in N_0$ or there exists an effective way to find a function φ, which is a composition of the functions θ_0, θ_1, θ_2 and $y \in N_0$ such that $\varphi(y) = x$.*

Lemma 21. *Let $\{A_a\}_{a\in Q}$ be a sequence of disjoint subsets of N_0. For any function φ, which is a composition of the functions θ_0, θ_1, θ_2, and for any $a \in Q$ there exists $b \in Q$ such that $\varphi([A_a]) \subseteq [A_b]$.*

Lemma 22. *Let $\{A_a\}_{a\in Q}$ be a sequence of disjoint non-empty subsets of N_0. For any function φ, which is a composition of the functions θ_0, θ_1, θ_2, and for any $a, b \in Q$ there exists an effective way to verify whether or not $\varphi([A_a]) \subseteq [A_b]$.*

Lemma 23. *Let $\{A_a\}_{a\in Q}$ be a sequence of disjoint non-empty subsets of N_0 and φ be a composition of the functions θ_0, θ_1, θ_2. If $a, b \in Q$ are such that $\varphi([A_a]) \subseteq [A_b]$, then there exist at least two different elements $c_1, c_2 \in Q$ such that $\varphi([A_{c_1}]) \subseteq [A_b]$ and $\varphi([A_{c_2}]) \subseteq [A_b]$.*

Theorem 3. *There exists an r.e. bc-degree, which contains different m-degrees of the type of $\mathbb{Q}$.*

Proof. The construction of such a degree is analogous to that in Theorem 1, i.e. we construct an r.e. sequence $\{A_a\}_{a\in Q}$ of disjoint subset of N_0 such that if $a \prec b$, then $[A_a] \leq_m [A_b]$, but $[A_b] \not\leq_m [A_a]$.

We construct the sets $\{A_a\}_{a\in Q}$ by steps, building the finite approximation $A_{a,s}$ of A_a, $a \in Q$, on step s.

On step s if $(s)_0 = \langle e, i, j, i_1, j_1\rangle$ and $(i, j) \prec (i_1, j_1)$, then our aim is to satisfy that the function Φ_e does not m-reduce $[A_{(i_1,j_1)}]$ to $[A_{(i,j)}]$, i.e. to find such a witness $x \in \mathrm{Dom}(\Phi_e)$ that at least one of the following two conditions is satisfied:

(i) $x \notin [A_{(i_1,j_1)}]$ & $\Phi_e(x) \in [A_{(i,j)}]$;

(ii) $x \in [A_{(i_1,j_1)}]$ & $\Phi_e(x) \notin [A_{(i,j)}]$.

Since the definitions are the same as in Theorem 1, we omit them and describe the construction of the sequence $\{A_a\}_{a\in Q}$.

Step $s = 0$. Let $N_2 = \{a_0 < a_1 < \ldots\}$; we take $A_{(i,j),0} = a_{\Pi(i,j)}$.

Step $s > 0$. If neither $[\mathrm{Seq}_5((s)_0)$ nor $\mathrm{Seq}_5((s)_0)\&(((s)_0)_1, ((s)_0)_2 + 1) \prec (((s)_0)_3, ((s)_0)_4 + 1)]$, then we do nothing, i.e. we take $A_{(i,j),s} = A_{(i,j),s-1}$, $i \in \mathbb{Z}, j \in \mathbb{N}$, and do not create any requirements.

If $\mathrm{Seq}_5((s)_0)$ and $s = \langle e, i, j, i_1, j_1\rangle$, where $(i,j) \prec (i_1, j_1)$, we verify whether an active $(s)_0$-requirement exists. If there exists such a requirement, then do nothing.

If such a requirement does not exist, then we verify whether there exists an $x \in N_1$ such that $x > r((s)_0)$, $x \in \mathrm{Dom}(\Phi_{e,s})$, $x \notin \cup_{a\in Q} A_{a,s-1}$ and x does not belong to any active negative requirement, created on a step $t < s$ such that $(t)_0 < (s)_0$. If such an x does not exist, then we do nothing.

Otherwise, we denote by x_s the least such x and create an $(s)_0$-requirement x_s. Let $\Phi_e(x_s) = z$ and $\psi(y) = z$, where ψ is either a composition of the functions $\{\varphi_k\}_{0\le k\le 2}$ or $\psi = \mathrm{id}$ and $y \in N_0$.

We verify whether $z \in A_{(i,j),s-1}$. If so, then we fix $A_{(i,j),s} = A_{(i,j),s-1} \cup \{x_s\}$, $A_{(k,l),s} = A_{(k,l),s-1}$ for $(k,l) \neq (i,j)$.

Otherwise, we verify if $z \in A_{(i',j'),s-1}$ for some $(i', j') \neq (i,j)$. If so, then fix $A_{(i_1,j_1),s} = A_{(i_1,j_1),s-1} \cup \{x_s\}$, $A_{(k,l),s} = A_{(k,l),s-1}$ for $(k,l) \neq (i_1, j_1)$. Otherwise, we consider two cases:

Case I. $x_s \neq y$. We fix $A_{(i_1,j_1),s} = A_{(i_1,j_1),s-1} \cup \{x_s\}$, $A_{(k,l),s} = A_{(k,l),s-1}$ for $(k,l) \neq (i_1, j_1)$ and create a negative e-requirement y.

Case II. $x_s = y$. We find effectively $(i_2, j_2) \neq (i_1, j_1)$ such that $\psi([A_{(i_2,j_2)}]) \subseteq [A_{(i,j)}]$ and fix $A_{(i_2,j_2),s} = A_{(i_2,j_2),s-1} \cup \{x_s\}$, $A_{(k,l),s} = A_{(k,l),s-1}$ for $(k,l) \neq (i_2, j_2)$.

Finally, we take $A_a = \cup_{s\in\mathbb{N}} A_{a,s}$, $a \in Q$.

Obviously, the construction is effective, hence the sequence $\{A_a\}_{a\in Q}$ is an r.e. sequence of disjoint subsets of N_0.

The proofs of the following lemmas are analogous to those in Theorem 1.

Lemma 24. *The set $N_1 \setminus A$ is infinite.*

Lemma 25. *The set $N_1 \setminus A$ is immune.*

Lemma 26. *For any natural number e, such that $N_1 \subseteq \mathrm{Dom}(\Phi_e)$, and for every $\langle e, i, j, i_1, j_1\rangle$, such that $(i,j) \prec (i_1, j_1)$, there exists a constant $\langle e, i, j, i_1, j_1\rangle$-requirement.*

Theorem 3 is completed.

Corollary 9. *There exists an r.e. tt-degree, which contains different m-degrees of the type of $\mathbb{Q}$.*

Combining the technique from Theorem 1 above and [3],Theorem 1 , one can receive that there exists an r.e. pc-degree, which contains infinite antichains of chains of the type of ω^k for different numbers k (but r.e.).

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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CO-SPECTRA OF JOINT SPECTRA OF STRUCTURES

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We introduce and study the notion of joint spectrum of finitely many abstract structures. A characterization of the lower bounds of the elements of the joint spectrum is obtained.

Keywords: enumeration reducibility, enumeration jump, enumeration degrees, forcing

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1. INTRODUCTION

Let $\mathfrak{A} = (\mathbb{N}; R_1, \ldots, R_k)$ be a structure with domain the set of all natural numbers $\mathbb{N}$, where each R_i is a subset of $\mathbb{N}^{r_i}$ and "=" and "$\neq$" are among $R_1, \ldots, R_k$.

An enumeration f of $\mathfrak{A}$ is a total mapping from $\mathbb{N}$ onto $\mathbb{N}$.

For every $A \subseteq \mathbb{N}^a$ define

$$f^{-1}(A) = \{\langle x_1 \ldots x_a \rangle : (f(x_1), \ldots, f(x_a)) \in A\}.$$

Let

$$f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \cdots \oplus f^{-1}(R_k).$$

For any sets of natural numbers A and B the set A is enumeration reducible to B ($A \leq_e B$) if there is an enumeration operator Γ_z such that $A = \Gamma_z(B)$. By $d_e(A)$ we denote the enumeration degree of the set A. The set A is total if $A \equiv_e A^+$, where $A^+ = A \oplus (\mathbb{N} \setminus A)$. An enumeration degree is called total if it contains a total set.

Definition 1.1. *The degree spectrum of* $\mathfrak{A}$ is the set

$$DS(\mathfrak{A}) = \{d_e(f^{-1}(\mathfrak{A})) : f \text{ is an enumeration of } \mathfrak{A}\}.$$

The notion is introduced by [6] for bijective enumerations. In [2, 5, 4, 7] several results about degree spectra of structures are obtained. In [7] it is shown that if $\mathbf{a} \in DS(\mathfrak{A})$ and $\mathbf{b}$ is a total e-degree, $\mathbf{a} \leq \mathbf{b}$, then $\mathbf{b} \in DS(\mathfrak{A})$. In other words, the degree spectrum of $\mathfrak{A}$ is closed upwards.

The co-spectrum of the structure $\mathfrak{A}$ is the set of all lower bounds of the degree spectra of $\mathfrak{A}$. Co-spectra are introduced and studied in [7].

The aim of the present paper is to study a generalization of the notions of degree spectra and co-spectra for finitely many structures and to give a normal form of the sets, which generates the elements of the generalized co-spectra in terms of recursive Σ^+ formulae.

In what follows we shall use the following Jump Inversion Theorem proved in [8]. Notice that the jump operation "$'$" denotes here the enumeration jump introduced by Cooper [3].

Given $n+1$ sets $B_0, \dots, B_n$, for every $i \leq n$ define the set $\mathcal{P}(B_0, \dots, B_i)$ by means of the following inductive definition:

(i) $\mathcal{P}(B_0) = B_0$;

(ii) If $i < n$, then $\mathcal{P}(B_0, \dots, B_{i+1}) = (\mathcal{P}(B_0, \dots, B_i))' \oplus B_{i+1}$.

Theorem 1.1. *Let $n > k \geq 0$, $B_0, \dots, B_n$ be arbitrary sets of natural numbers. Let $A \subseteq \mathbb{N}$ and let Q be a total subset of $\mathbb{N}$ such that $\mathcal{P}(B_0, \dots, B_n) \leq_e Q$ and $A^+ \leq_e Q$. Suppose also that $A \not\leq_e \mathcal{P}(B_0, \dots, B_k)$. Then there exists a total set F having the following properties:*

(i) *For all $i \leq n$, $B_i \leq_e F^{(i)}$;*

(ii) *For all $i, 1 \leq i \leq n$, $F^{(i)} \equiv_e F \oplus \mathcal{P}(B_0, \dots, B_{i-1})'$;*

(iii) *$F^{(n)} \equiv_e Q$;*

(iv) *$A \not\leq_e F^{(k)}$.*

2. JOINT SPECTRA OF STRUCTURES

Let us fix the structures $\mathfrak{A}_0, \dots, \mathfrak{A}_n$.

Definition 2.1. *The joint spectrum of* $\mathfrak{A}_0, \dots, \mathfrak{A}_n$ is the set

$$DS(\mathfrak{A}_0, \mathfrak{A}_1, \dots, \mathfrak{A}_n) = \{\mathbf{a} : \mathbf{a} \in DS(\mathfrak{A}_0), \mathbf{a}' \in DS(\mathfrak{A}_1), \dots, \mathbf{a}^{(\mathbf{n})} \in DS(\mathfrak{A}_n)\}.$$

Definition 2.2. Let $k \leq n$. *The k-th jump spectrum of* $\mathfrak{A}_0, \dots, \mathfrak{A}_n$ is the set

$$DS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n) = \{\mathbf{a}^{(\mathbf{k})} : \mathbf{a} \in DS(\mathfrak{A}_0, \dots, \mathfrak{A}_n)\}.$$

Proposition 2.1. *$DS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$ is closed upwards, i.e. if $\mathbf{a}^{(\mathbf{k})} \in DS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$, $\mathbf{b}$ is a total e-degree and $\mathbf{a}^{(\mathbf{k})} \leq \mathbf{b}$, then $\mathbf{b} \in DS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$.*

Proof. Suppose that $\mathbf{a}^{(\mathbf{k})} \in DS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$, $\mathbf{b}$ is a total degree and $\mathbf{b} \geq \mathbf{a}^{(\mathbf{k})}$. By the Jump Inversion Theorem 1.1 there is a total e-degree $\mathbf{f}$ such that:

(1) $\mathbf{a^{(i)}} \leq \mathbf{f^{(i)}}$ for all $i \leq k$;
(2) $\mathbf{f^{(k)}} = \mathbf{b}$.

Clearly, $\mathbf{a^{(i)}} \leq \mathbf{f^{(i)}}$ for $i \leq n$. Since $\mathbf{a^{(i)}} \in DS(\mathfrak{A}_i)$ and $\mathbf{f^{(i)}}$ is total, $\mathbf{f^{(i)}} \in DS(\mathfrak{A}_i)$, $i \leq n$. Therefore $\mathbf{f} \in DS(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$ and hence $\mathbf{b} = \mathbf{f^{(k)}} \in DS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$. □

Definition 2.3. Let $k \leq n$. *The k-th co-spectrum of* $\mathfrak{A}_0, \dots, \mathfrak{A}_n$ is the set of all lower bounds of $DS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$, i.e.

$$CS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n) = \{\mathbf{b} : \mathbf{b} \in \mathcal{D}_e \& (\forall \mathbf{a} \in DS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n))(\mathbf{b} \leq \mathbf{a})\}.$$

Proposition 2.2. *Let $k \leq n$. Then*

$$CS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_k, \dots, \mathfrak{A}_n) = CS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_k).$$

Proof. Clearly, $DS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_k, \dots, \mathfrak{A}_n) \subseteq DS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_k)$ and hence

$$CS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_k) \subseteq CS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_k, \dots, \mathfrak{A}_n).$$

To show the reverse inclusion, let $\mathbf{c} \in CS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$, i.e. $\mathbf{c} \leq \mathbf{a^{(k)}}$ for all $\mathbf{a} \in DS(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$. Suppose that $\mathbf{c} \notin CS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_k)$. Then there exist sets C and A such that $d_e(C) = \mathbf{c}$ and $d_e(A) \in DS(\mathfrak{A}_0, \dots, \mathfrak{A}_k)$ and $C \not\leq_e A^{(k)}$. Notice that $\mathcal{P}(A, A', \dots, A^{(k)}) \equiv_e A^{(k)}$ and therefore $C \not\leq_e \mathcal{P}(A, A', \dots, A^{(k)})$. Fix some sets $B_1, \dots, B_{n-k}$ such that $d_e(B_1) \in DS(\mathfrak{A}_{k+1}), \dots, d_e(B_{n-k}) \in DS(\mathfrak{A}_n)$. Applying the Jump Inversion Theorem 1.1, we obtain a total set F such that:

(i) For all $i \leq k$, $A^{(i)} \leq_e F^{(i)}$;
(ii) For all $j, 1 \leq j \leq n-k$, $B_j \leq_e F^{(k+j)}$;
(iii) $C \not\leq_e F^{(k)}$.

Since the degree spectra are closed upwards, $d_e(F^{(i)}) \in DS(\mathfrak{A}_i)$, $i = 0, \dots, n$, and hence $d_e(F) \in DS(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$. On the other hand, $C \not\leq_e F^{(k)}$ and hence $c \notin CS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$. A contradiction. □

Theorem 2.1. *Let $A \subseteq \mathbb{N}$. Then the following are equivalent:*

(1) $d_e(A) \in CS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_k)$.
(2) *For every $k+1$ enumerations $f_0, \dots, f_k$,*

$$A \leq_e \mathcal{P}(f_0^{-1}(\mathfrak{A}_0), \dots, f_k^{-1}(\mathfrak{A}_k))).$$

Proof. Suppose that A satisfies (2) and consider a $\mathbf{b} \in DS(\mathfrak{A}_0, \dots, \mathfrak{A}_k)$. We shall show that $d_e(A) \leq \mathbf{b^{(k)}}$.

Let $i \leq k$. Then $\mathbf{b^{(i)}} \in DS(\mathfrak{A}_i)$ and hence there exists an enumeration f_i such that $\mathbf{b^{(i)}} = d_e(f_i^{-1}(\mathfrak{A}_i))$. Clearly, $d_e(A) \leq d_e(\mathcal{P}(f_0^{-1}(\mathfrak{A}_0), \dots, f_k^{-1}(\mathfrak{A}_k))) = \mathbf{b^{(k)}}$.

Suppose now that $d_e(A) \in CS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_k)$ and $f_0, \dots, f_k$ are enumerations. Set $B_i = f_i^{-1}(\mathfrak{A}_i)$, $i = 0, \dots, k$. Towards a contradiction assume that $A \not\leq_e \mathcal{P}(f_0^{-1}(\mathfrak{A}_0), \dots, f_k^{-1}(\mathfrak{A}_k))$. By the Jump Inversion Theorem 1.1 there is a total set F such that: $B_i \leq_e F^{(i)}$, $i \leq k$, and $A \not\leq_e F^{(k)}$. Clearly, $d_e(F) \in DS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_k)$ and $d_e(A) \not\leq F^{(k)}$. So, $d_e(A) \notin CS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_k)$. A contradiction. □

3. GENERIC ENUMERATIONS AND FORCING

3.1. THE SATISFACTION RELATION

Given $k+1$ enumerations $f_0, \ldots, f_k$, denote by $\bar{f}$ the sequence $f_0, \ldots, f_k$ and set for $i \leq k$, $\mathcal{P}_i^{\bar{f}} = \mathcal{P}(f_0^{-1}(\mathfrak{A}_0), \ldots, f_i^{-1}(\mathfrak{A}_i))$.

Let $W_0, \ldots, W_z, \ldots$ be a Gödel enumeration of the r.e. sets and D_v be the finite set having canonical code v.

For every $i \leq k$, e and x in $\mathbb{N}$ define the relations $\bar{f} \models_i F_e(x)$ and $\bar{f} \models_i \neg F_e(x)$ by induction on i:

(i) $\bar{f} \models_0 F_e(x) \iff (\exists v)(\langle v, x\rangle \in W_e \;\&\; D_v \subseteq f_0^{-1}(\mathfrak{A}_0))$;

(ii)
$$\begin{aligned} \bar{f} \models_{i+1} F_e(x) \iff (\exists v)(\langle v, x\rangle \in W_e \;\&\; (\forall u \in D_v)(&\\ u = \langle 0, e_u, x_u\rangle \;\&\; \bar{f} \models_i F_{e_u}(x_u) \vee&\\ u = \langle 1, e_u, x_u\rangle \;\&\; \bar{f} \models_i \neg F_{e_u}(x_u) \vee&\\ u = \langle 2, x_u\rangle \;\&\; x_u \in f_{i+1}^{-1}(\mathfrak{A}_{i+1})))&; \end{aligned}$$

(iii) $\bar{f} \models_i \neg F_e(x) \iff \bar{f} \not\models_i F_e(x)$.

From the above definition follows easily the truth of the following

Proposition 3.1. *Let $A \subseteq \mathbb{N}$ and $i \leq k$. Then*
$$A \leq_e \mathcal{P}_i^{\bar{f}} \iff (\exists e)(A = \{x : \bar{f} \models_i F_e(x)\}).$$

3.2. FINITE PARTS AND FORCING

The forcing conditions, which we shall call *finite parts*, are k-tuples $\bar{\tau} = (\tau_0, \ldots, \tau_k)$ of finite mappings $\tau_0, \ldots, \tau_k$ of $\mathbb{N}$ in $\mathbb{N}$. We shall use the letters $\bar{\delta}, \bar{\tau}, \bar{\rho}, \bar{\mu}$ to denote finite parts.

For every $i \leq k$, e and x in $\mathbb{N}$ and every finite part $\bar{\tau}$ we define the forcing relations $\bar{\tau} \Vdash_i F_e(x)$ and $\bar{\tau} \Vdash_i \neg F_e(x)$, following the definition of relations "$\models_i$".

Definition 3.1.

(i) $\bar{\tau} \Vdash_0 F_e(x) \iff (\exists v)(\langle v, x\rangle \in W_e \;\&\; D_v \subseteq \tau_0^{-1}(\mathfrak{A}_0))$;

(ii)
$$\begin{aligned} \bar{\tau} \Vdash_{i+1} F_e(x) \iff \exists v(\langle v, x\rangle \in W_e \;\&&\\ (\forall u \in D_v)(u = \langle 0, e_u, x_u\rangle \;\&\; \bar{\tau} \Vdash_i F_{e_u}(x_u) \vee&\\ u = \langle 1, e_u, x_u\rangle \;\&\; \bar{\tau} \Vdash_i \neg F_{e_u}(x_u) \vee&\\ u = \langle 2, x_u\rangle \;\&\; x_u \in \tau_{i+1}^{-1}(\mathfrak{A}_{i+1})))&; \end{aligned}$$

(iii) $\bar{\tau} \Vdash_i \neg F_e(x) \iff (\forall \bar{\rho} \supseteq \bar{\tau})(\bar{\rho} \not\Vdash_i F_e(x))$.

Given finite parts $\bar{\delta} = (\delta_0, \ldots, \delta_k)$ and $\bar{\tau} = (\tau_0, \ldots, \tau_k)$, let
$$\bar{\delta} \subseteq \bar{\tau} \iff \delta_0 \subseteq \tau_0, \ldots, \delta_k \subseteq \tau_k.$$

Proposition 3.2. *Let $i \leq k, e, x \in \mathbb{N}$ and $\bar{\delta} = (\delta_0, \ldots, \delta_k)$, $\bar{\tau} = (\tau_0, \ldots, \tau_k)$ be finite parts :*

(1) $\bar{\delta} \subseteq \bar{\tau}$, *then* $\bar{\delta} \Vdash_i (\neg)F_e(x) \Longrightarrow \bar{\tau} \Vdash_i (\neg)F_e(x)$;
(2) *If* $\delta_0 = \tau_0, \ldots, \delta_i = \tau_i$, *then* $\bar{\delta} \Vdash_i (\neg)F_e(x) \iff \bar{\tau} \Vdash_i (\neg)F_e(x)$.

Proof. The monotonicity condition (1) is obvious.

The proof of (2) is by induction on i. Skipping the obvious case $i = 0$, suppose that $i < k$ and

$$\bar{\delta} \Vdash_i (\neg)F_e(x) \iff \bar{\tau} \Vdash_i (\neg)F_e(x).$$

Let $\tau_j = \delta_j, j \leq i+1$. From the definition of the relation $\Vdash_{i+1}$ it follows immediately that

$$\bar{\delta} \Vdash_{i+1} F_e(x) \iff \bar{\tau} \Vdash_{i+1} F_e(x).$$

Assume that $\bar{\delta} \Vdash_{i+1} \neg F_e(x)$, but $\bar{\tau} \nVdash_{i+1} \neg F_e(x)$. Then there exists a finite part $\bar{\rho} \supseteq \bar{\tau}$ such that $\bar{\rho} \Vdash_{i+1} F_e(x)$. Consider the finite part $\bar{\mu}$ such that $\mu_j = \rho_j$ for $j \leq i+1$, and $\mu_j = \delta_j$ for $i+1 < j \leq k$. Clearly, $\bar{\mu} \supseteq \bar{\delta}$ and $\bar{\mu} \Vdash_{i+1} F_e(x)$. A contradiction. □

Definition 3.2. If $\bar{\delta} = (\delta_0, \ldots, \delta_k)$, $\bar{\tau} = (\tau_0, \ldots, \tau_k)$ and $i \leq k$, define

$$\bar{\delta} \subseteq_i \bar{\tau} \iff \delta_0 \subseteq \tau_0, \ldots, \delta_i \subseteq \tau_i, \delta_{i+1} = \tau_{i+1}, \ldots, \delta_k = \rho_k.$$

Let $\bar{\tau} \Vdash_i^* (\neg)F_e(x)$ be the same as $\bar{\tau} \Vdash_i (\neg)F_e(x)$ with the exception of

(iii) $\bar{\tau} \Vdash_i \neg F_e(x) \iff (\forall \bar{\rho} \supseteq_i \bar{\tau})(\bar{\rho} \nVdash_i^* F_e(x))$.

As an immediate corollary of the previous proposition, we get the following

Lemma 3.1. *For each* $i \leq k, e, x \in \mathbb{N}$ *and* $\bar{\tau}$,

$$\bar{\tau} \Vdash_i (\neg)F_e(x) \iff \bar{\tau} \Vdash_i^* (\neg)F_e(x).$$

3.3. GENERIC ENUMERATIONS

For any $i \leq k, e, x \in \mathbb{N}$ denote by $X^i_{\langle e,x \rangle} = \{\bar{\rho} : \bar{\rho} \Vdash_i F_e(x)\}$.

If $\bar{f} = (f_0, \ldots, f_k)$ is an enumeration of $\mathfrak{A}_0, \ldots, \mathfrak{A}_k$, then

$$\bar{\tau} \subseteq \bar{f} \iff \tau_0 \subseteq f_0, \ldots, \tau_k \subseteq f_k.$$

Definition 3.3. An enumeration $\bar{f}$ of $\mathfrak{A}_0, \ldots, \mathfrak{A}_k$ is *i-generic* if for every $j < i$, $e, x \in \mathbb{N}$,

$$(\forall \bar{\tau} \subseteq \bar{f})(\exists \bar{\rho} \in X^j_{\langle e,x \rangle})(\bar{\tau} \subseteq \bar{\rho}) \Longrightarrow (\exists \bar{\tau} \subseteq \bar{f})(\bar{\tau} \in X^j_{\langle e,x \rangle}).$$

Lemma 3.2. (1) *Let* $\bar{f}$ *be an i-generic enumeration. Then*

$$\bar{f} \models_i F_e(x) \iff (\exists \bar{\tau} \subseteq \bar{f})(\bar{\tau} \Vdash_i F_e(x)).$$

(2) *Let* f *be an* $(i+1)$*-generic enumeration. Then*

$$\bar{f} \models_i \neg F_e(x) \iff (\exists \bar{\tau} \subseteq \bar{f})(\bar{\tau} \Vdash_i \neg F_e(x)).$$

Proof. Induction on i. Clearly, for every $\bar{f}$ we have

$$\bar{f} \models_0 F_e(x) \iff (\exists \bar{\tau} \subseteq \bar{f})(\bar{\tau} \Vdash_0 F_e(x)).$$

From the definition of the relations $\models_i$ and $\Vdash_i$ it follows immediately that if for some enumeration $\bar{f}$ we have the equivalences

$$\bar{f} \models_i F_e(x) \iff (\exists \bar{\tau} \subseteq \bar{f})(\bar{\tau} \Vdash_i F_e(x))$$

and

$$\bar{f} \models_i \neg F_e(x) \iff (\exists \bar{\tau} \subseteq \bar{f})(\bar{\tau} \Vdash_i \neg F_e(x)),$$

then we have also

$$\bar{f} \models_{i+1} F_e(x) \iff (\exists \bar{\tau} \subseteq \bar{f})(\bar{\tau} \Vdash_{i+1} F_e(x)).$$

So, to finish the proof, we have to show that if for some $i < k$ the enumeration $\bar{f}$ is $(i+1)$-generic and (1) holds, then (2) holds as well. Indeed, suppose that $\bar{f} \models_i \neg F_e(x)$. Assume that there is no $\bar{\tau} \subseteq \bar{f}$ such that $\bar{\tau} \Vdash_i \neg F_e(x)$. Then for every $\bar{\tau} \subseteq \bar{f}$ there exists a finite part $\bar{\rho} \supseteq \bar{\tau}$ such that $\bar{\rho} \Vdash_i F_e(x)$. From the $(i+1)$-genericity of $\bar{f}$ it follows that there exists a finite part $\bar{\tau} \subseteq \bar{f}$ such that $\bar{\tau} \Vdash_i F_e(x)$. Hence $\bar{f} \models_i F_e(x)$. A contradiction.

Assume now that $\bar{\tau} \subseteq \bar{f}$ and $\bar{\tau} \Vdash_i \neg F_e(x)$. Assume that $\bar{f} \models_i F_e(x)$. Then we can find a finite part $\bar{\mu} \subseteq \bar{f}$ such that $\bar{\mu} \Vdash_i F_e(x)$ and $\bar{\mu} \supseteq \bar{\tau}$. A contradiction. □

3.4. FORCING K-DEFINABLE SETS

Definition 3.4. The set $A \subseteq \mathbb{N}$ is *forcing k-definable* on $\mathfrak{A}_0, \ldots, \mathfrak{A}_k$ if there exist a finite part $\bar{\delta}$ and $e \in \mathbb{N}$ such that

$$x \in A \iff (\exists \bar{\tau} \supseteq \bar{\delta})(\bar{\tau} \Vdash_k F_e(x)).$$

Theorem 3.1. *Let $A \subseteq \mathbb{N}$. If $A \leq_e \mathcal{P}(f_0^{-1}(\mathfrak{A}_0), \ldots, f_k^{-1}(\mathfrak{A}_k))$ for all $f_0, \ldots, f_k$ enumerations of $\mathfrak{A}_0, \ldots, \mathfrak{A}_k$, respectively, then A is forcing k-definable on $\mathfrak{A}_0, \ldots, \mathfrak{A}_k$.*

Proof. Suppose that A is not forcing k-definable on $\mathfrak{A}_0, \ldots, \mathfrak{A}_k$.

We shall construct a $(k+1)$-generic enumeration $\bar{f}$ such that $A \not\leq \mathcal{P}_k^{\bar{f}}$.

The construction of the enumeration $\bar{f}$ will be carried out by steps. On each step j we shall define a finite part $\bar{\delta}^j = (\delta_0^j, \ldots, \delta_k^j)$, so that $\bar{\delta}^j \subseteq \bar{\delta}^{j+1}$, and take $f_i = \cup_j \delta_i^j$ for each $i \leq k$.

On the steps $j = 3q$ we shall ensure that each f_i is a total surjective mapping from $\mathbb{N}$ onto $\mathbb{N}$. On the steps $j = 3q+1$ we shall ensure that $\bar{f}$ is $(k+1)$-generic. On the steps $j = 3q+2$ we shall ensure that $A \not\leq \mathcal{P}_k^{\bar{f}}$.

Let $\bar{\delta}^0 = (\emptyset, \ldots, \emptyset)$. Suppose that $\bar{\delta}^j$ is defined.

Case $j = 3q$. For every i, $0 \leq i \leq k$, let x_i be the least natural number, which does not belong to the domain of δ_i^j, and y_i be the least natural number, which does not belong to the range of δ_i^j. Let $\delta_i^{j+1}(x_i) = y_i$ and $\delta_i^{j+1}(x) \simeq \delta_i^j(x)$ for $x \neq x_i$.

Case $j = 3\langle e, i, x\rangle + 1$, $i \leq k$. Check if there exists a finite part $\bar{\rho} \supseteq \bar{\delta}^j$ such that $\bar{\rho} \Vdash_i F_e(x)$. If so, then let $\bar{\delta}^{j+1}$ be the least such ρ. Otherwise let $\bar{\delta}^{j+1} = \bar{\delta}^j$.

CASE $j = 3q + 2$. Consider the set

$$C = \{x : (\exists \bar{\tau} \supseteq \bar{\delta}^j)(\bar{\tau} \Vdash_k F_q(x))\}.$$

Clearly, C is forcing k-definable on $\mathfrak{A}_0, \ldots, \mathfrak{A}_k$ and hence $C \neq A$. Then there exists an x such that either $x \in A$ and $x \notin C$ or $x \in C$ and $x \notin A$. Take $\bar{\delta}^{j+1} = \bar{\delta}^j$ in the first case.

If the second case holds, then there must exist a $\bar{\rho} \supseteq \bar{\delta}^j$ such that $\bar{\rho} \Vdash_k F_q(x)$. Let $\bar{\delta}^{j+1}$ be the least such $\bar{\rho}$.

Let $\bar{\delta}^{j+1} = \bar{\delta}^j$ in the other cases.

To prove that the so received enumeration $\bar{f} = \cup_j \bar{\delta}^j$ is $(k+1)$-generic, let us fix numbers $i \leq k$, $e, x \in \mathbb{N}$ and suppose that for every finite part $\bar{\tau} \subseteq \bar{f}$ there is an extention $\bar{\rho} \Vdash_i F_e(x)$. Then consider the step $j = 3\langle e, i, x\rangle + 1$. From the construction we have that $\bar{\delta}^{j+1} \Vdash_i F_e(x)$.

Suppose there is a $q \in \mathbb{N}$, so that $A = \{x : \bar{f} \models_k F_q(x)\}$. Consider the step $j = 3q + 2$. From the construction there is an x such that one of the following two cases holds:

(a) $x \in A$ and $(\forall \bar{\rho} \supseteq \bar{\delta}^j)(\bar{\rho} \nVdash_k F_q(x))$. So, $\bar{\delta}^j \Vdash_k \neg F_q(x)$. Since $\bar{f}$ is $(k+1)$-generic, $x \in A$ & $\bar{f} \not\models_k F_q(x)$. A contradiction.

(b) $x \notin A$ & $\bar{\delta}^{j+1} \Vdash_k F_q(x)$. Since $\bar{f}$ is $(k+1)$-generic, $\bar{f} \models_k F_q(x)$. A contradiction. □

4. THE NORMAL FORM THEOREM

In this section we shall give an explicit form of the forcing k-definable on $\mathfrak{A}_0$, $\ldots$, $\mathfrak{A}_k$ sets by means of *positive* recursive Σ^+_k formulae. These formulae can be considered as a modification of Ash's formulae introduced in [1].

4.1. RECURSIVE Σ^+_K FORMULAE

Let, for each $i \leq k$, $\mathfrak{L}_i = \{T^i_1, \ldots, T^i_{n_i}\}$ be the language of $\mathfrak{A}_i$, where every T^i_j is an r^i_j-ary predicate symbol, and $\mathfrak{L} = \mathfrak{L}_0 \cup \cdots \cup \mathfrak{L}_k$. We suppose that the languages $\mathfrak{L}_0, \ldots, \mathfrak{L}_k$ are disjoint.

For each $i \leq k$ fix a sequence $\mathbb{X}^i_0, \ldots, \mathbb{X}^i_n, \ldots$ of variables. The upper index i in the variable $\mathbb{X}^i_j$ shows that the possible values of $\mathbb{X}^i_j$ will be in $|\mathfrak{A}_i|$. By $\bar{X}^i$ we shall denote finite sequences of variables of the form $X^i_0, \ldots, X^i_l$.

For each $i \leq k$, define the elementary Σ^+_i formulae and the Σ^+_i formulae by induction on i, as follows.

Definition 4.1.

(1) *An elementary Σ^+_0 formula* with free variables among $\bar{X}^0$ is an existential formula of the form

$$\exists Y^0_1 \ldots \exists Y^0_m \Phi(\bar{X}^0, Y^0_1, \ldots, Y^0_m),$$

where Φ is a finite conjunction of atomic formulae in $\mathfrak{L}_0$ with variables among $Y^0_1, \ldots, Y^0_m, \bar{X}^0$;

(2) *An elementary* Σ^+_{i+1} *formula* with free variables among $\bar{X}^0, \dots, \bar{X}^{i+1}$ is in the form

$$\exists \bar{Y}^0 \dots \exists \bar{Y}^{i+1} \Phi(\bar{X}^0, \dots, \bar{X}^{i+1}, \bar{Y}^0, \dots, \bar{Y}^{i+1}),$$

where Φ is a finite conjunction of Σ^+_i formulae and negations of Σ^+_i formulae with free variables among $\bar{Y}^0, \dots, \bar{Y}^i, \bar{X}^0, \dots, \bar{X}^i$ and atoms of $\mathcal{L}_{i+1}$ with variables among $\bar{X}^{i+1}, \bar{Y}^{i+1}$;

(3) *A* Σ^+_i *formula* with free variables among $\bar{X}^0, \dots, \bar{X}^i$ is an r.e. infinitary disjunction of elementary Σ^+_i formulae with free variables among $\bar{X}^0, \dots, \bar{X}^i$.

Let Φ be a Σ^+_i formula, $i \le k$, with free variables among $\bar{X}^0, \dots, \bar{X}^i$ and let $\bar{t}^0, \dots, \bar{t}^i$ be elements of $\mathbb{N}$. Then by $(\mathfrak{A}_0, \dots, \mathfrak{A}_i) \models \Phi(\bar{X}^0/\bar{t}^0, \dots, \bar{X}^i/\bar{t}^i)$ we shall denote that Φ is true on $(\mathfrak{A}_0, \dots, \mathfrak{A}_i)$ under the variable assignment v such that $v(\bar{X}^0) = \bar{t}^0, \dots, v(\bar{X}^i) = \bar{t}^i$. More precisely, we have the following

Definition 4.2.

(1) If $\Phi = \exists Y^0_1 \dots \exists Y^0_m \Psi(\bar{X}^0, Y^0_1, \dots, Y^0_m)$ is a Σ^+_0 formula, then

$$(\mathfrak{A}_0) \models \Phi(\bar{X}^0/\bar{t}^0) \iff \exists s_1 \dots \exists s_m (\mathfrak{A}_0 \models \Psi(\bar{X}^0/\bar{t}^0, Y^0_1/s_1, \dots, Y^0_m/s_m)).$$

(2) If $\Phi = \exists \bar{Y}^0 \dots \exists \bar{Y}^{i+1} \Psi(\bar{X}^0, \dots, \bar{X}^{i+1}, \bar{Y}^0, \dots, \bar{Y}^{i+1})$ and $\Psi = (\varphi \ \& \ \alpha)$, where $\varphi(\bar{X}^0, \dots, \bar{X}^i, \bar{Y}^0, \dots, \bar{Y}^i)$ is a conjunction of Σ^+_i formulae and negations of Σ^+_i formulae and $\alpha(\bar{Y}^{i+1}, \bar{X}^{i+1})$ is a conjuction of atoms of $\mathcal{L}_{i+1}$, then

$$(\mathfrak{A}_0, \dots, \mathfrak{A}_{i+1}) \models \Phi(\bar{X}^0/\bar{t}^0, \dots, \bar{X}^{i+1}/\bar{t}^{i+1}) \iff$$
$$\exists \bar{s}^0 \dots \exists \bar{s}^{i+1} ((\mathfrak{A}_0, \dots, \mathfrak{A}_i) \models \varphi(\bar{X}^0/\bar{t}^0, \dots, \bar{X}^i/\bar{t}^i, \bar{Y}^0/\bar{s}^0, \dots, \bar{Y}^i/\bar{s}^i) \ \&$$
$$(\mathfrak{A}_{i+1}) \models \alpha(\bar{X}^{i+1}/\bar{t}^{i+1}, \bar{Y}^{i+1}/\bar{s}^{i+1})).$$

4.2. THE FORMALLY K-DEFINABLE SETS

Definition 4.3. The set $A \subseteq \mathbb{N}$ is *formally k-definable* on $\mathfrak{A}_0, \dots, \mathfrak{A}_k$ if there exists a recursive sequence $\{\Phi\}^{\gamma(x)}$ of Σ^+_k formulae with free variables among $\bar{W}^0, \dots, \bar{W}^k$ and elements $\bar{t}^0, \dots, \bar{t}^k$ of $\mathbb{N}$ such that the following equivalence holds:

$$x \in A \iff (\mathfrak{A}_0 \dots \mathfrak{A}_k) \models \Phi^{\gamma(x)}(\bar{W}^0/\bar{t}^0, \dots, \bar{W}^k/\bar{t}^k).$$

We shall show that every forcing k-definable set is formally k-definable.

Let for every i, $0 \le i \le k$, var_i be an effective bijective mapping of the natural numbers onto the variables with upper index i. Given a natural number x, by X^i we shall denote the variable $var_i(x)$.

Let $y_1 < y_2 < \dots < y_k$ be the elements of a finite set D, let Q be one of the quantifiers $\exists$ or $\forall$, and let Φ be an arbitrary formula. Then by $Q^i(y : y \in D)\Phi$ we shall denote the formula $QY^i_1 \dots QY^i_k \Phi$.

Proposition 4.1. *Let $\bar{E} = (E_0, \ldots, E_k)$ be a sequence of finite sets of natural numbers, where $E_j = \{w_0^j, \ldots, w_{s_j}^j\}$. Let $i \leq k, x, e$ be elements of $\mathbb{N}$. There exists an uniform effective way to construct a Σ_i^+ formula $\Phi^i_{\bar{E},e,x}$ with free variables among $\bar{W}^0, \ldots, \bar{W}^k$, where $W_j^i = var(w_j^i)$, such that for every finite part $\bar{\delta} = (\delta_0, \ldots, \delta_k)$,* $\mathrm{dom}(\delta_0) = E_0, \ldots, \mathrm{dom}(\delta_k) = E_k$,

$$(\mathfrak{A}_0, \ldots, \mathfrak{A}_k) \models \Phi^i_{\bar{E},e,x}(\bar{W}^0/\delta_0(\bar{w}^0), \ldots, \bar{W}^k/\delta_k(\bar{w}^k)) \iff \bar{\delta} \Vdash_i^* F_e(x).$$

Proof. We shall construct the formula $\Phi^i_{\bar{E},e,x}$ by induction on i following the definition of the forcing.

(1) Let $i = 0$. Let $V = \{v : \langle v, x\rangle \in W_e\}$. Consider an element v of V. For every $u \in D_v$ define the atom Π_u as follows:

(a) If $u = \langle j, x_1^0, \ldots, x_{r_j}^0\rangle$, where $1 \leq j \leq n_0$ and all $x_1^0, \ldots, x_{r_j}^0$ are elements of E_0, then let $\Pi_u = T_j^0(X_1^0, \ldots, X_{r_j}^0)$;

(b) Let $\Pi_u = X_0^0 \neq X_0^0$ in the other cases.

Set $\Pi_v = \bigwedge_{u \in D_v} \Pi_u$ and $\Phi^0_{\bar{E},e,x} = \bigvee_{v\in V} \Pi_v$.

(2) Case $i + 1$. Let $V = \{v : \langle v, x\rangle \in W_e\}$ and $v \in V$.

For every $u \in D_v$ define the formula Π_u as follows:

(a) If $u = \langle 0, e_u, x_u\rangle$, then let $\Pi_u = \Phi^i_{\bar{E},e_u,x_u}$;

(b) If $u = \langle 1, e_u, x_u\rangle$, then let

$$\Pi_u = \neg[\bigvee_{E_0^* \supseteq E_0 \ldots E_i^* \supseteq E_i} (\exists^0 y \in E_0^* \setminus E_0) \ldots (\exists^i y \in E_i^* \setminus E_i) \Phi^i_{\bar{E}^*,e_u,x_u}],$$

where $\bar{E}^* = (E_0^*, \ldots, E_i^*, E_{i+1}, \ldots, E_k)$;

(c) If $u = \langle 2, x_u\rangle$, $x_u = \langle j, x_1^{i+1}, \ldots, x_{r_j}^{i+1}\rangle$, $j \leq n_{i+1}$ and $x_1^{i+1}, \ldots, x_{r_j}^{i+1} \in E_{i+1}$, then let $\Pi_u = T_j^{i+1}(X_1^{i+1}, \ldots, X_{r_j}^{i+1})$;

(d) Let $\Pi_u = \Phi^i_{\{\emptyset\}.0.0} \wedge \neg\Phi^i_{\{\emptyset\},0,0}$ in the other cases.

Now let $\Pi_v = \bigwedge_{u\in D_v} \Pi_u$ and set $\Phi^{i+1}_{\bar{E},e,x} = \bigvee_{v\in V} \Pi_v$. An induction on i shows that for every i the Σ_i^+ formula $\Phi^i_{\bar{E},e,x}$ satisfies the requirements of the proposition.□

Theorem 4.1. *Let $A \subseteq \mathbb{N}$ be forcing k-definable on $\mathfrak{A}_0, \ldots, \mathfrak{A}_k$. Then A is formally k-definable on $\mathfrak{A}_0, \ldots, \mathfrak{A}_k$.*

Proof. If A is forcing k-definable on $\mathfrak{A}_0, \ldots, \mathfrak{A}_k$, then there exist a finite part $\bar{\delta} = (\delta_0, \ldots, \delta_k)$ and $e \in \mathbb{N}$ such that

$$x \in A \iff (\exists \bar{\tau} \supseteq \bar{\delta})(\bar{\tau} \Vdash_k F_e(x)) \iff (\exists \bar{\tau} \supseteq \bar{\delta})(\bar{\tau} \Vdash_k^* F_e(x)).$$

Let for $i = 1, \ldots, k$, $E_i = \mathrm{dom}(\delta_i) = \{w_1^i, \ldots, w_r^i\}$ and let $\delta(w_j^i) = t_j^i$, $j = 1, \ldots, r$. Set $\bar{E} = (E_0, \ldots, E_k)$. From the previous proposition we know that

$$(\mathfrak{A}_0, \ldots, \mathfrak{A}_k) \models \bigvee_{\bar{E}^* \supseteq \bar{E}} \exists(y \in \bar{E}^* \setminus \bar{E})\Phi^k_{\bar{E}^*,e,x}(\bar{W}^0/\bar{t}^0, \ldots, \bar{W}^k/\bar{t}^k) \iff$$

$$(\exists \bar{\tau} \supseteq \bar{\delta})(\mathrm{dom}(\bar{\tau}) = \bar{E}^*)(\bar{\tau} \Vdash_k^* F_e(x)).$$

Then for all $x \in \mathbb{N}$ the following equivalence is true:

$$x \in A \iff (\mathfrak{A}_0, \dots, \mathfrak{A}_k) \models \bigvee_{\bar{E}^* \supseteq \bar{E}} \exists(y \in \bar{E}^* \setminus \bar{E}) \Phi^k_{\bar{E}^*, e, x}(\bar{W}^0/\bar{t}^0, \dots, \bar{W}^k/\bar{t}^k).$$

From here we can conclude that A is formally k-definable on $\mathfrak{A}_0, \dots, \mathfrak{A}_k$. □

Theorem 4.2. *Let $A \subseteq \mathbb{N}$. Then the following are equivalent:*

(1) $d_e(A) \in CS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$, $k \leq n$.

(2) *For every enumeration $\bar{f}$ of* $\mathfrak{A}_0, \dots, \mathfrak{A}_k$, $A \leq_e \mathcal{P}(f_0^{-1}(\mathfrak{A}_0), \dots, f_k^{-1}(\mathfrak{A}_k))$.

(3) *A is forcing k-definable on* $\mathfrak{A}_0, \dots, \mathfrak{A}_k$.

(4) *A is formally k-definable on* $\mathfrak{A}_0, \dots, \mathfrak{A}_k$.

Proof. The equivalence (1) $\iff$ (2) follows from Theorem 2.1.

The implication (2) $\Rightarrow$ (3) follows from Theorem 3.1.

The implication (3) $\Rightarrow$ (4) follows from the previous theorem.

The last implication (4) $\Rightarrow$ (2) follows by induction on i. □

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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DEGREE SPECTRA AND CO-SPECTRA OF STRUCTURES

IVAN N. SOSKOV

Given a countable structure $\mathfrak{A}$, we define the degree spectrum $DS(\mathfrak{A})$ of $\mathfrak{A}$ to be the set of all enumeration degrees generated by the presentations of $\mathfrak{A}$ on the natural numbers. The co-spectrum of $\mathfrak{A}$ is the set of all lower bounds of $DS(\mathfrak{A})$. We prove some general properties of the degree spectra, which show that they behave with respect to their co-spectra very much like the cones of enumeration degrees. Among the results are the analogs of Selman's Theorem [14], the Minimal Pair Theorem and the existence of a quasi-minimal enumeration degree.

Keywords: degree spectra, enumeration degrees

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1. INTRODUCTION

Given a countable abstract structure $\mathfrak{A}$, we define the degree spectrum $DS(\mathfrak{A})$ of $\mathfrak{A}$ to be the set of all enumeration degrees generated by the presentations of $\mathfrak{A}$ on the natural numbers. The co-spectrum of $\mathfrak{A}$ is the set of all lower bounds of $DS(\mathfrak{A})$. As a typical example of a spectrum one may consider the cone of the total degrees, greater than or equal to some $\mathbf{a}$, and the respective co-spectrum which is equal to the set of all degrees less than or equal to $\mathbf{a}$. There are examples of structures with more complicated degree spectra, e.g. [11, 8, 2, 7, 15]. In any case the co-spectrum of a structure is a countable ideal and as we shall see, every countable ideal can be represented as co-spectrum of some structure.

Here we shall prove some general properties of the degree spectra, which show that the degree spectra behave with respect to their co-spectra very much like the cones of enumeration degrees. Among the results we would like to mention the analogs of Selman's Theorem [14], the Minimal Pair Theorem and the existence of a quasi-minimal enumeration degree. These results are known in two versions in the theory of the enumeration degrees – above one fixed degree and above a sequence of degrees, while our approach gives a unified treatment of both cases. Another possible benefit is that the objects constructed in the proofs are elements of the degree spectra or closely related to them, which gives an additional information about their complexity.

Finally, our results pose some restrictions on the sets of degrees, which can be represented as degree spectra. For example, using the existence of quasi-minimal degrees, we obtain that if a degree spectrum posseses a countable base of total degrees, then it has a least element. As a consequence of this, we get that for every two incomparable Turing degrees **a** and **b** there does not exist a structure $\mathfrak{A}$ such that $DS(\mathfrak{A})$ is equal to the union of the cones above **a** and **b**, answering negatively a question apparently posed by Goncharov.

2. PRELIMINARIES

2.1. ORDINAL NOTATIONS

In what follows we shall consider only recursive ordinals α, which are below a fixed recursive ordinal η. We shall suppose that a notation $e \in \mathcal{O}$ for η is fixed and the notations for the ordinals $\alpha < \eta$ are elements a of $\mathcal{O}$ such that $a <_o e$. For the definitions of the set $\mathcal{O}$ and the relation "$<_o$" the reader may consult [12] or [13]. We shall identify every ordinal with its notation and denote the ordinals by the letters α, β, γ and δ. In particular, we shall write $\alpha < \beta$ instead of $\alpha <_o \beta$. If α is a limit ordinal, then by $\{\alpha(p)\}_{p\in\mathbb{N}}$ we shall denote the unique strongly increasing sequence of ordinals with limit α, determined by the notation of α, and write $\alpha = \lim \alpha(p)$.

2.2. ENUMERATION DEGREES

Let A and B be sets of natural numbers. Then A is *enumeration reducible* to B, $A \leq_e B$, if $A = \Gamma_z(B)$ for some enumeration operator Γ_z. In other words, using the notation D_v for the finite set having canonical code v, and $W_0, \ldots, W_z, \ldots$ for the Gödel enumeration of the r.e. sets, we have

$$A \leq_e B \iff \exists z \forall x (x \in A \iff \exists v(\langle v, x\rangle \in W_z \ \& \ D_v \subseteq B)).$$

The relation $\leq_e$ is reflexive and transitive and induces an equivalence relation $\equiv_e$ on all subsets of $\mathbb{N}$. The respective equivalence classes are called enumeration degrees. We shall denote by $d_e(A)$ the enumeration degree containing A and by $\mathcal{D}_e = (\mathcal{D}_e, \leq, \mathbf{0}_e)$ the structure of the enumeration degrees, where " $\leq$ " is the partial ordering on $\mathcal{D}_e$, induced by " $\leq_e$ ", and $\mathbf{0}_e$ is the least enumeration degree consisting of all recursively enumerable sets. For an introduction to the enumeration degrees the reader might consult Cooper ([6]).

Given a set A of natural numbers, denote by A^+ the set $A \oplus (\mathbb{N} \setminus A)$. The set A is called *total* iff $A \equiv_e A^+$. An enumeration degree is total if it contains a total set. The substructure $\mathcal{D}_T$ of $\mathcal{D}_e$, consisting of all total degrees, is isomorphic to the structure of the Turing degrees. Therefore we may identify the Turing degrees with the total enumeration degrees.

The enumeration jump operator is defined in Cooper [5] and further studied by McEvoy [10]. Here we shall use the following definition of the e-jump, which is m-equivalent to the original one, see [10]:

Definition 2.1. Given a set A, let $K_A^0 = \{\langle x, z\rangle : x \in \Gamma_z(A)\}$. Define the *e-jump* A' of A to be the set $(K_A^0)^+$.

The following properties of the enumeration jump are proved in [10]:

Let A and B be sets of natural numbers. Set $B^{(0)} = B$ and $B^{(n+1)} = (B^{(n)})'$.

(J1) If $A \leq_e B$, then $A' \leq_e B'$.

(J2) A is Σ^0_{n+1} relatively to B iff $A \leq_e (B^+)^{(n)}$.

Given an enumeration degree $\mathbf{a} = d_e(A)$, let for every natural number n, $\mathbf{a}^{(n)} = d_e(A^{(n)})$. Notice that the jump is well defined on all enumeration degrees and that it is consistent with the Turing jump on the total enumeration degrees.

For every recursive ordinal α the α-th iteration of the enumeration jump $\mathbf{a}^{(\alpha)}$ is defined in a way similar to that one used in the definition of the α-th iteration of the Turing jump, see [17]. Again it turns out that both definitions are consistent on the total enumeration degrees.

2.3. DEGREE SPECTRA

We shall consider structures of the kind $\mathfrak{A} = (\mathbb{N}; R_1, \dots, R_k)$, where " $=$ " and " $\neq$ " are among $R_1, \dots, R_k$.

Enumeration of $\mathfrak{A}$ is every total surjective mapping of $\mathbb{N}$ onto $\mathbb{N}$.

Given an enumeration f of $\mathfrak{A}$ and a subset of A of $\mathbb{N}^a$, let

$$f^{-1}(A) = \{\langle x_1, \dots, x_a\rangle : (f(x_1), \dots, f(x_a)) \in A\}.$$

By $f^{-1}(\mathfrak{A})$ we shall denote the set $f^{-1}(R_1) \oplus \cdots \oplus f^{-1}(R_k)$. In particular, if $f = \lambda x.x$, then $f^{-1}(\mathfrak{A})$ will be denoted by $D(\mathfrak{A})$.

Definition 2.2. *The degree spectrum of* $\mathfrak{A}$ is the set

$$DS(\mathfrak{A}) = \{d_e(f^{-1}(\mathfrak{A})) : f \text{ is an enumeration of } \mathfrak{A})\}.$$

If $\mathbf{a}$ is the least element of $DS(\mathfrak{A})$, then $\mathbf{a}$ is called the *degree of* $\mathfrak{A}$.

The notion of degree spectrum is introduced in [11], where the first results about degrees of structures are obtained. In [8] Knight defines the so-called jump degrees of structures:

Definition 2.3. Let $\alpha < \omega_1^{CK}$. Then the α-th jump spectrum of $\mathfrak{A}$ is the set

$$DS_\alpha = \{d_e(f^{-1}(\mathfrak{A})^{(\alpha)}) : f \text{ is an enumeration of } \mathfrak{A}\}.$$

If $\mathbf{a}$ is the least element of DS_α, then $\mathbf{a}$ is called the *α-th jump degree of* $\mathfrak{A}$.

There are two main differences between the standard definition of the notion of degree spectrum of a structure considered in [11] and [8] and the one introduced here.

First of all, in the cited papers the pullback $f^{-1}(\mathfrak{A})$ of a structure is defined by taking into account not only the positive part of the predicates, but also the negative one. So the degree spectrum in the sense of [11] and [8] is equal to $DS(\mathfrak{A}^+)$, where

$$\mathfrak{A}^+ = (\mathbb{N}, R_1, \dots, R_k, \neg R_1, \dots, \neg R_k).$$

It can be easily seen that $DS(\mathfrak{A}^+)$ consists only of total enumeration degrees. We shall call structures of that kind *total*. More precisely,

Definition 2.4. A structure $\mathfrak{A}$ is *total* if all elements of $DS(\mathfrak{A})$ are total.

The second difference is connected to the enumerations. In [11] and [8] the degree spectra are defined by taking into account only the bijective enumerations, while we allow arbitrary surjective enumerations. The choice of the class of enumerations reflects on the notion of degree spectrum of a given structure. For example, let $\mathfrak{A} = (\mathbb{N}; =, \neq)$. Clearly, if we define the degree spectrum of $\mathfrak{A}$ by taking into account only the bijective enumerations, then it will be equal to $\{\mathbf{0}_e\}$, while if we take all surjective enumerations, then $DS(\mathfrak{A})$ will consist of all total enumeration degrees. Fortunately, this difference does not affect the notion of degree of a structure. Namely, the following Proposition is true:

Proposition 2.1. *Let f be an arbitrary enumeration of $\mathfrak{A}$. There exists a bijective enumeration g of $\mathfrak{A}$ such that $g^{-1}(\mathfrak{A}) \leq_e f^{-1}(\mathfrak{A})$.*

Proof. Let $E_f = \{\langle x, y\rangle : f(x) = f(y)\}$. Clearly, $E_f^+ \leq_e f^{-1}(\mathfrak{A})$. Define the function h by means of primitive recursion as follows:

$$\begin{aligned} h(0) &\simeq 0, \\ h(n+1) &\simeq \mu z[(\forall k \leq n)(\langle h(k), z\rangle \notin E_f)]. \end{aligned}$$

Set $g(n) = f(h(n))$. Now one can easily check that g is bijective and $g^{-1}(\mathfrak{A}) \oplus E_f^+ \equiv_e f^{-1}(\mathfrak{A})$. □

The main benefit of defining $DS(\mathfrak{A})$ by taking all surjective enumerations is that it is always closed upwards with respect to the total enumeration degrees:

Proposition 2.2. *Let g be an enumeration of $\mathfrak{A}$. Suppose that F is a total set and $g^{-1}(\mathfrak{A}) \leq_e F$. There exists an enumeration f of $\mathfrak{A}$ such that $f^{-1}(\mathfrak{A}) \equiv_e F$.*

Proof. Fix two distinct elements s and t of $\mathbb{N}$. Define the mapping $f(x)$ as follows:

$$f(x) \simeq \begin{cases} g(x/2), & \text{if } x \text{ is even}, \\ s, & \text{if } x = 2z+1 \text{ and } z \in F, \\ t, & \text{if } x = 2z+1 \text{ and } z \notin F. \end{cases}$$

Since "=" and "≠" are among the underlined predicates of $\mathfrak{A}$, we have that $F \leq_e f^{-1}(\mathfrak{A})$.

To prove that $f^{-1}(\mathfrak{A}) \leq_e F$, consider the predicate R_i of $\mathfrak{A}$. Let us fix two natural numbers x_s and x_t such that $g(x_s) \simeq s$ and $g(x_t) \simeq t$. Let $x_1, \dots, x_{r_i}$ be arbitrary natural numbers. Define the natural numbers $y_1, \dots, y_{r_i}$ by means of the following recursive in F procedure. Let $1 \leq j \leq r_i$. If x_j is even, then let $y_j x_j/2$. If $x_j = 2z + 1$ and $z \in F$, then let $y_j = x_s$. If $x_j = 2z + 1$ and $z \notin F$, then let $y_j = x_t$. Clearly,

$$\langle x_1, \dots, x_{r_i}\rangle \in f^{-1}(R_i) \iff \langle y_1, \dots, y_{r_i}\rangle \in g^{-1}(R_i).$$

Since $g^{-1}(\mathfrak{A}) \leq_e F$, from the last equivalence it follows that $f^{-1}(R_i) \leq_e F$. So we obtain that $f^{-1}(\mathfrak{A}) \leq_e F$. □

Remark. The requirement that the set F is total is necessary for the truth of the proposition. Indeed, if the structure $\mathfrak{A}$ were total, then for all enumerations f of $\mathfrak{A}$ the set $f^{-1}(\mathfrak{A})$ would be total.

The results in [11] show that there exist structures, e.g. linear orderings, which do not posses degrees. Further investigations in [8, 2, 7] show that for every recursive ordinal α there exist linear orderings with α-th jump degree $\mathbf{0}^{(\alpha)}$, which do not possess β-th jump degree for $\beta < \alpha$.

3. CO-SPECTRA OF STRUCTURES

Definition 3.1. Let $\mathcal{A}$ be a set of enumeration degrees, the *co-set of* $\mathcal{A}$ is the set $co(\mathcal{A})$ of all lower bounds of $\mathcal{A}$. Namely,

$$co(\mathcal{A}) = \{b : b \in \mathcal{D}_e \ \& \ (\forall a \in \mathcal{A})(b \leq_e a)\}.$$

The co-set of the α-th jump spectrum of a structure $\mathfrak{A}$ will be called α-th jump co-spectrum of $\mathfrak{A}$ and will be denoted by $CS_\alpha(\mathfrak{A})$. In particular, if $\alpha = 0$, the set $CS_\alpha(\mathfrak{A})$ will be denoted by $CS(\mathfrak{A})$ and called co-spectrum of $\mathfrak{A}$.

Evidently, for every $\mathcal{A} \subseteq \mathcal{D}_e$ the set $co(\mathcal{A})$ is a countable ideal. As we shall see later, every countable ideal can be represented as a co-spectrum of some structure $\mathfrak{A}$.

Definition 3.2. Let $A \subseteq \mathbb{N}$, $\alpha < \omega_1^{CK}$ and let f be an enumeration of $\mathfrak{A}$. The set A is called α-*admissible* in the enumeration f if $A \leq_e f^{-1}(\mathfrak{A})^{(\alpha)}$.

The set A is α-admissible in $\mathfrak{A}$ if A is admissible in all enumerations of $\mathfrak{A}$.

Clearly, an enumeration degree $\mathbf{a}$ belongs to $CS_\alpha(\mathfrak{A})$ iff $\mathbf{a}$ contains an α-admissible set. Our close goal is to show that the α-admissible sets admit a characterization in terms of the structure $\mathfrak{A}$. Thus we shall obtain some information about the elements of $CS_\alpha(\mathfrak{A})$. Our characterization is a generalization of the one presented in [3], where only total structures are considered. Another reason for presenting this characterization here is that we want to obtain an upper bound of the degrees in $DS_\alpha(\mathfrak{A})$, which determine the elements of $CS_\alpha(\mathfrak{A})$.

Let us fix a structure $\mathfrak{A} = (\mathbb{N}; R_1, \dots, R_k)$.

In what follows, we shall use the term "finite part" to denote arbitrary finite mappings of $\mathbb{N}$ into $\mathbb{N}$. The finite parts will be denoted by δ, τ, ρ, etc.

Definition 3.3. Let $\alpha < \omega_1^{CK}$. An enumeration f of $\mathfrak{A}$ is α-generic if for every $\beta < \alpha$ and for every set S of finite parts such that $S \leq_e D(\mathfrak{A})^{(\beta)}$ the following condition holds:

$$(\exists \tau \subseteq f)(\tau \in S \vee (\forall \rho \supseteq \tau)(\rho \notin S)).$$

Proposition 3.1. *Suppose that $\alpha < \omega_1^{CK}$ and let f be an α-generic enumeration. Then for every $\beta < \alpha$, $f^{-1}(\mathfrak{A}) \not\leq_e D(\mathfrak{A})^{(\beta)}$ and hence $f^{-1}(\mathfrak{A})^{(\beta)} \not\leq_e D(\mathfrak{A})^{(\beta)}$.*

Proof. Let $\beta < \alpha$. Consider the set $\bar{E} = \{\langle x, y\rangle : f(x) \neq f(y)\}$. Clearly, $\bar{E} \leq_e f^{-1}(\mathfrak{A})^{(\beta)}$. Assume that $f^{-1}(\mathfrak{A}) \leq_e D(\mathfrak{A})^{(\beta)}$. Then the set

$$S = \{\tau : (\exists x, y \in Dom(\tau))(\langle x, y\rangle \in \bar{E} \;\&\; \tau(x) \simeq \tau(y))\}$$

is enumeration reducible to $D(\mathfrak{A})^{(\beta)}$ and hence there exists a $\tau \subseteq f$ such that $\tau \in S$ or $(\forall \rho \supseteq \tau)(\rho \notin S)$. Evidently, both conditions are impossible. □

Corollary 3.1. *If f is an α-generic enumeration, then $d_e(f^{-1}(\mathfrak{A})^{(\beta)})$ does not belong to $CS_\beta(\mathfrak{A})$ for any $\beta < \alpha$.*

For every α, e and x in $\mathbb{N}$ we define the relations $f \models_\alpha F_e(x)$ and $f \models_\alpha \neg F_e(x)$ as follows:

(i) $f \models_0 F_e(x)$ iff there exists a v such that $\langle v, x\rangle \in W_e$ and for all $u \in D_v$

$$(\exists i)(1 \leq i \leq k \;\&\; u = \langle i, x_1^u, \dots, x_{r_i}^u\rangle \;\&\; (f(x_1^u), \dots, f(x_{r_i}^u)) \in R_i);$$

(ii) Let $\alpha = \beta + 1$. Then

$$\begin{aligned} f \models_\alpha F_e(x) \iff (\exists v)(\langle v, x\rangle \in W_e \;\&\; (\forall u \in D_v)(\\ (u = \langle 0, e_u, x_u\rangle \;\&\; f \models_\beta F_{e_u}(x_u)) \vee \\ (u = \langle 1, e_u, x_u\rangle \;\&\; f \models_\beta \neg F_{e_u}(x_u)))); \end{aligned}$$

(iii) Let $\alpha = \lim \alpha(p)$. Then

$$\begin{aligned} f \models_\alpha F_e(x) \iff (\exists v)(\langle v, x\rangle \in W_e \;\&\; (\forall u \in D_v)(\\ u = \langle p_u, e_u, x_u\rangle \;\&\; f \models_{\alpha(p_u)} F_{e_u}(x_u))); \end{aligned}$$

(iv) $f \models_\alpha \neg F_e(x) \iff f \not\models_\alpha F_e(x)$.

An immediate corollary of the definitions above is the following:

Lemma 3.1. *Let $A \subseteq \mathbb{N}$ and let $\alpha < \omega_1^{CK}$. Then $A \leq_e f^{-1}(\mathfrak{A})^{(\alpha)}$ iff there exists an e such that $A = \{x : f \models_\alpha F_e(x)\}$.*

For every $\alpha < \omega_1^{CK}$, e and x in $\mathbb{N}$ and every finite part τ we define the forcing relations $\tau \Vdash_\alpha F_e(x)$ and $\tau \Vdash_\alpha \neg F_e(x)$, following the definition of "$\models$":

(i) $\tau \Vdash_0 F_e(x)$ iff there exists a v such that

$\langle v, x\rangle \in W_e$ and for all $u \in D_v$, $u = \langle i, x_1^u, \dots, x_{r_i}^u\rangle, 1 \leq i \leq k$,

$$x_1^u, \dots, x_{r_i}^u \in \mathrm{dom}(\tau) \;\&\; (\tau(x_1^u), \dots, \tau(x_{r_i}^u)) \in R_i;$$

(ii) Let $\alpha = \beta + 1$. Then

$$\tau \Vdash_\alpha F_e(x) \iff (\exists v)(\langle v, x\rangle \in W_e \ \&\ (\forall u \in D_v)($$
$$(u = \langle 0, e_u, x_u\rangle \ \&\ \tau \Vdash_\beta F_{e_u}(x_u)) \vee$$
$$(u = \langle 1, e_u, x_u\rangle \ \&\ \tau \Vdash_\beta \neg F_{e_u}(x_u))));$$

(iii) Let $\alpha = \lim \alpha(p)$. Then

$$\tau \Vdash_\alpha F_e(x) \iff (\exists v)(\langle v, x\rangle \in W_e \ \&\ (\forall u \in D_v)($$
$$u = \langle p_u, e_u, x_u\rangle \ \&\ \tau \Vdash_{\alpha(p_u)} F_{e_u}(x_u)));$$

(iv) $\tau \Vdash_\alpha \neg F_e(x) \iff (\forall \rho \supseteq \tau)(\rho \nVdash_\alpha F_e(x))$.

For every recursive ordinal α, $e, x \in \mathbb{N}$ set $X^\alpha_{\langle e,x\rangle} = \{\rho : \rho \Vdash_\alpha F_e(x)\}$.

Given a sequence $\{X_n\}$ of sets of natural numbers, say that $\{X_n\}$ is *e-reducible* to the set P if there exists a recursive function g such that for all n we have $X_n = \Gamma_{g(n)}(P)$. The sequence $\{X_n\}$ is T-reducible to P if the function $\lambda n, x.\chi_{X_n}(x)$ is recursive in P.

From the definition of the enumeration jump it follows immediately that if $\{X_n\}$ is e-reducible to P, then $\{X_n\}$ is T-reducible to P'.

Lemma 3.2. *For every α the sequence $\{X^\alpha_n\}$ is uniformly in α e-reducible to $f^{-1}(\mathfrak{A})^{(\alpha)}$, and hence it is uniformly in α T-reducible to $f^{-1}(\mathfrak{A})^{(\alpha+1)}$.*

Proof. Using effective transfinite recursion and following the definition of the forcing, one can define a recursive function $g(\alpha, n)$ such that for every α, $X^\alpha_n = \Gamma_{g(\alpha,n)}(f^{-1}(\mathfrak{A})^{(\alpha)})$. □

The next properties of the forcing relation follow easily from the definitions and the previous lemma:

Lemma 3.3. (1) *Let α be a recursive ordinal, $e, x \in \mathbb{N}$, and let $\tau \subseteq \rho$ be finite parts. Then*

$$\tau \Vdash_\alpha (\neg)F_e(x) \Rightarrow \rho \Vdash_\alpha (\neg)F_e(x).$$

(2) *Let f be an α-generic enumeration. Then*

$$f \models_\alpha F_e(x) \iff (\exists \tau \subseteq f)(\tau \Vdash_\alpha F_e(x)).$$

(3) *Let f be an $(\alpha+1)$-generic enumeration. Then*

$$f \models_\alpha \neg F_e(x) \iff (\exists \tau \subseteq f)(\tau \Vdash_\alpha \neg F_e(x)).$$

Definition 3.4. Let $A \subseteq \mathbb{N}$ and let α be a recursive ordinal. The set A is *forcing α-definable* on $\mathfrak{A}$ if there exist a finite part δ and $e, x \in \mathbb{N}$ such that

$$A = \{x : (\exists \tau \supseteq \delta)(\tau \Vdash_\alpha F_e(x))\}.$$

Clearly, if A is forcing α-definable on $\mathfrak{A}$, then $A \leq_e f^{-1}(\mathfrak{A})^{(\alpha)}$. The vice versa is not always true. As we shall see later, the forcing α-definable sets coincide with the sets which are α-admissible in $\mathfrak{A}$.

The next proposition follows easily from the definitions:

Proposition 3.2. *Let $\mathfrak{B} = (\mathbb{N}, R'_1, \ldots, R'_k)$ be a structure isomorphic to $\mathfrak{A}$ and α be a recursive ordinal. Then every forcing α-definable on $\mathfrak{B}$ set is forcing α-definable on $\mathfrak{A}$.*

Proposition 3.3. *Let α be a recursive ordinal, $\beta < \alpha$ and let $A \subseteq \mathbb{N}$ be not forcing β-definable on $\mathfrak{A}$. There exists an α-generic enumeration f of $\mathfrak{A}$ satisfying the following conditions:*

(1) $f \leq_e A^+ \oplus D(\mathfrak{A})^{(\alpha)}$;
(2) *If* $\gamma \leq \alpha$, *then* $f^{-1}(\mathfrak{A})^{(\gamma)} \leq_e f \oplus D(\mathfrak{A})^{(\gamma)}$;
(3) $A \not\leq_e f^{-1}(\mathfrak{A})^{(\beta)}$.

Proof. We shall construct the enumeration f by steps. At each step q we shall define a finite part δ_q, so that $\delta_q \subseteq \delta_{q+1}$, and take $f = \bigcup_q \delta_q$. We shall consider three kinds of steps. At steps $q = 3r$ we shall ensure that the mapping f is total and surjective. At steps $q = 3r + 1$ we shall ensure that f is $(\alpha + 1)$-generic and at steps $q = 3r + 2$ we shall ensure that f satisfies (3).

Let S denote the set of all finite parts. If $\alpha = \xi + 1$, then for every natural number n set $Y_n = \Gamma_n(D(\mathfrak{A})^{(\xi)}) \cap S$. If $\alpha = \lim \alpha(p)$ is a limit ordinal, then set $Y_n = \Gamma_{(n)_0}(D(\mathfrak{A})^{(\alpha((n)_1))}) \cap S$.

In both cases we have that the sequence $\{Y_n\}$ is T-reducible to $D(\mathfrak{A})^{(\alpha)}$ and consists of all sets S of finite parts which are enumeration reducible to $D(\mathfrak{A})^{(\gamma)}$ for some $\gamma < \alpha$.

Let δ_0 be the empty finite part and suppose that δ_q is defined.

a) Case $q = 3r$. Let x_0 be the least natural number which does not belong to $\mathrm{dom}(\delta_q)$ and let s_0 be the least natural number which does not belong to the range of δ_q. Set $\delta_{q+1}(x_0) \simeq s_0$ and $\delta_{q+1}(x) \simeq \delta_q(x)$ for $x \neq x_0$.

b) Case $q = 3r + 1$. Consider the set Y_r.

Check whether there exists an element ρ of Y_r such that $\delta_q \subseteq \rho$. If the answer is positive, then let δ_{q+1} be the least extension of δ_q belonging to Y_r. If the answer is negative, then let $\delta_{q+1} = \delta_q$.

c) Case $q = 3r + 2$. Consider the set

$$C_r = \{x : (\exists \tau \supseteq \delta_q)(\tau \Vdash_\beta F_r(x))\}.$$

Clearly, C_r is forcing β-definable on $\mathfrak{A}$ and hence $C_r \neq A$. Notice that $C_r \leq_e D(\mathfrak{A})^\beta$ uniformly in r and δ_q. Therefore the set C_r is recursive in $D(\mathfrak{A})^{(\alpha)}$ uniformly in r and δ_q. Let x_r be the least natural number such that

$$x_r \in C_r \ \& \ x_r \notin A \vee x_r \notin C_r \ \& \ x_r \in A.$$

Suppose that $x_r \in C_r$. Then there exists a τ such that

$$\delta_q \subseteq \tau \ \& \ \tau \Vdash_\beta F_r(x_r). \tag{3.1}$$

Let δ_{q+1} be the least τ satisfying (3.1). If $x_r \notin C_r$, then set $\delta_{q+1} = \delta_q$. Notice that in this case we have $\delta_{q+1} \Vdash_\beta \neg F_r(x_r)$.

From the construction above it follows immediately that $f = \bigcup_q \delta_q$ is e-reducible to $A^+ \oplus D(\mathfrak{A})^{(\alpha)}$ and hence it satisfies (1).

Let $\gamma \leq \alpha$. Then there exists an e such that $f^{-1}(\mathfrak{A})^{(\gamma)} = \{x : f \models_\gamma F_e(x)\}$. Since f is α-generic, we can rewrite the last equality as $f^{-1}(\mathfrak{A})^{(\gamma)} = \{x : (\exists \tau \subseteq f)(\tau \Vdash_\gamma F_e(x))\}$. Therefore $f^{-1}(\mathfrak{A})^{(\gamma)} \leq_e f \oplus D(\mathfrak{A})^{(\gamma)}$.

It remains to show that $A \not\leq_e f^{-1}(\mathfrak{A})^{(\beta)}$. Towards a contradiction assume that $A \leq_e f^{-1}(\mathfrak{A})^{(\beta)}$. Then there exists an r such that

$$A = \{x : f \models_\beta F_r(x)\}.$$

Consider the step $q = 3r + 2$. By the construction we have

$$x_r \notin A \;\&\; \delta_{q+1} \Vdash_\beta F_r(x_r) \vee x_r \in A \;\&\; \delta_{q+1} \Vdash_\beta \neg F_r(x_r).$$

Hence by the genericity of f

$$x_r \notin A \;\&\; f \models_\beta F_r(x_r) \vee x_r \in A \;\&\; f \models_\beta \neg F_r(x_r).$$

A contradiction. □

Repeating the proof above without bothering about the set A, we get also the following:

Proposition 3.4. *Let α be a recursive ordinal. Then there exists an α-generic enumeration f such that f and $f^{-1}(\mathfrak{A})^{(\alpha)}$ are enumeration reducible to $D(\mathfrak{A})^{(\alpha)}$.*

Theorem 3.1. *Let α be a recursive ordinal, $\beta < \alpha$ and let $A \subseteq \mathbb{N}$ be not forcing β-definable on $\mathfrak{A}$. Let Q be a total set such that $A^+ \oplus D(\mathfrak{A})^{(\alpha)} \leq_e Q$. Then there exists an enumeration f satisfying the following conditions :*

(1) *The enumeration degree of $f^{-1}(\mathfrak{A})$ is total;*
(2) $A \not\leq_e f^{-1}(\mathfrak{A})^{(\beta)}$;
(3) $f^{-1}(\mathfrak{A})^{(\alpha)} \equiv_e Q$.

Proof. According Proposition 3.3 there exists an enumeration g of $\mathfrak{A}$ such that $g \leq_e Q$, $g^{-1}(\mathfrak{A})^{(\alpha)} \leq_e Q$ and $A \not\leq_e g^{-1}(\mathfrak{A})^{(\beta)}$.

From Jump Inversion Theorem [17] it follows that there exists a total set F such that the following assertions are true:

(i) $g^{-1}(\mathfrak{A}) \leq_e F$;
(ii) $A \not\leq_e F^{(\beta)}$;
(iv) $F^{(\alpha)} \equiv_e Q$.

By Proposition 2.2 there exists an enumeration f such that $f^{-1}(\mathfrak{A}) \equiv_e F$. □

Definition 3.5. Let Q be a total subset of $\mathbb{N}$ and $\alpha < \omega_1^{CK}$. An enumeration f of $\mathfrak{A}$ is *α, Q-acceptable* if f satisfies the following conditions:

(i) The enumeration degree of $f^{-1}(\mathfrak{A})$ is total;
(ii) $f^{-1}(\mathfrak{A})^{(\alpha)} \equiv_e Q$.

Theorem 3.2. *Let α be a recursive ordinal, $\beta < \alpha$ and let $A \subseteq \mathbb{N}$ be not forcing definable on $\mathfrak{A}$. Consider an enumeration g and a total set $Q \geq_e g^{-1}(\mathfrak{A})^{(\alpha)} \oplus A^+$. There exists an α, Q-acceptable enumeration f of $\mathfrak{A}$ such that $A \not\leq_e f^{-1}(\mathfrak{A})^{(\beta)}$.*

Proof. According Proposition 2.1 there exists a bijective enumeration h such that $h^{-1}(\mathfrak{A}) \leq_e g^{-1}(\mathfrak{A})$. Denote by $\mathfrak{B}$ the structure $(\mathbb{N}; h^{-1}(R_1), \dots, h^{-1}(R_k))$. Clearly, A is not β-forcing definable on $\mathfrak{B}$ and $D(\mathfrak{B}) \equiv_e h^{-1}(\mathfrak{A})$. Hence $D(\mathfrak{B})^{(\alpha)} \leq_e Q$. Let i be an enumeration such that the enumeration degree of $i^{-1}(\mathfrak{B})$ is total. $i^{-1}(\mathfrak{B})^{(\alpha)} \equiv_e Q$ and $A \not\leq_e i^{-1}(\mathfrak{B})^{(\beta)}$. Set $f = \lambda x.h(i(x))$. Then $f^{-1}(\mathfrak{A}) \equiv_e i^{-1}(\mathfrak{B})$. Thus f is α, Q-acceptable and $A \not\leq_e f^{-1}(\mathfrak{A})^{(\beta)}$. □

Corollary 3.2. *For every total $Q \geq_e g^{-1}(\mathfrak{A})^{(\alpha)}$ there exists an α, Q-acceptable enumeration of $\mathfrak{A}$.*

Theorem 3.3. *Let α be a constructive ordinal and $A \subseteq \mathbb{N}$. Let $\beta < \alpha$. Consider an enumeration g of $\mathfrak{A}$. Suppose that $Q \geq_e g^{-1}(\mathfrak{A})^{(\alpha)}$, Q is a total set and for all α, Q-acceptable enumerations f of $\mathfrak{A}$ we have $A \leq_e f^{-1}(\mathfrak{A})^{(\beta)}$. Then A is forcing β-definable on $\mathfrak{A}$.*

Proof. First we shall show that $A^+ \leq_e Q$. Clearly, there exists an enumeration h of $\mathfrak{A}$ such that h is α, Q-acceptable. Then $A \leq_e h^{-1}(\mathfrak{A})^{(\beta)}$. By the monotonicity of the enumeration jump we can conclude that

$$A' \leq_e h^{-1}(\mathfrak{A})^{(\alpha)} \leq_e Q.$$

Since $A^+ \leq_e A'$, we get that $A^+ \leq_e Q$.

Assume that A is not forcing α-definable on $\mathfrak{A}$. Applying Theorem 3.2, we obtain an α, Q-acceptable enumeration f such that $A \not\leq_e f^{-1}(\mathfrak{A})^{(\alpha)}$. A contradiction.

3.2. NORMAL FORM OF THE FORCING DEFINABLE SETS

In this subsection we shall show that the forcing definable sets on the structure $\mathfrak{A}$ coincide with the sets which are definable on $\mathfrak{A}$ by means of a certain kind of *positive* recursive Σ^0_α formulae. This formulae can be considered as a modification of the formulae introduced in [1], which is appropriate for their use on abstract structures.

Let $\mathfrak{L} = \{T_1, \dots, T_k\}$ be the first order language corresponding to the structure $\mathfrak{A}$. So, every T_i is an r_i-ary predicate symbol. We shall suppose also fixed a sequence $\mathbb{X}_0, \dots, \mathbb{X}_n, \dots$ of variables. The variables will be denoted by the letters X, Y, W, possibly indexed.

Next we define for $\alpha < \omega_1^{CK}$ the Σ^+_α formulae. The definition is by transfinite recursion on α and goes along with the definition of indices (codes) for every formula. We shall leave to the reader the explicit definition of the indices of our formulae, which can be done in a natural way.

Definition 3.6.

(i) Let $\alpha = 0$. The elementary Σ^+_α formulae are formulae in prenex normal form with a finite number of existential quantifiers and a matrix which is a finite conjunction of atomic predicates built up from the variables and the predicate symbols $T_1, \dots, T_k$.

(ii) Let $\alpha = \beta + 1$. An elementary Σ_α^+ formula is in the form

$$\exists Y_1 \dots \exists Y_m M(X_1, \dots, X_l, Y_1, \dots, Y_m),$$

where M is a finite conjunction of atoms of Σ_β^+ formulae and negations of Σ_β^+ formulae with free variables among $X_1, \dots, X_l$, $Y_1, \dots, Y_m$.

(iii) Let $\alpha = \lim \alpha(p)$ be a limit ordinal. The elementary Σ_α^+ formulae are in the form

$$\exists Y_1 \dots \exists Y_m M(X_1, \dots, X_l, Y_1, \dots, Y_m),$$

where M is a finite conjunction of $\Sigma_{\alpha(p)}^+$ formulae with free variables among $X_1, \dots, X_l, Y_1, \dots, Y_m$.

(iv) A Σ_α^+ formula with free variables among $X_1, \dots, X_l$ is an r.e. infinitary disjunction of elementary Σ_α^+ formulae with free variables among $X_1, \dots, X_l$.

Notice that the Σ_α^+ formulae are effectively closed under existential quantification and infinitary r.e. disjunctions.

Let Φ be a Σ_α^+ formula with free variables among $W_1, \dots, W_n$ and let $t_1, \dots, t_n$ be elements of $\mathbb{N}$. Then by $\mathfrak{A} \models \Phi(W_1/t_1, \dots, W_n/t_n)$ we shall denote that Φ is true on $\mathfrak{A}$ under the variable assignment v such that $v(W_1) = t_1, \dots, v(W_n) = t_n$.

Definition 3.7. Let $A \subseteq \mathbb{N}$ and let α be a constructive ordinal. The set A is *formally α-definable* on $\mathfrak{A}$ if there exists a recursive function $g(x)$ taking values indeces of Σ_α^+ formulae $\Phi_{g(x)}$ with free variables among $W_1, \dots, W_r$ and elements $t_1, \dots, t_r$ of $\mathbb{N}$ such that for every element x of $\mathbb{N}$ the following equivalence holds:

$$x \in A \iff \mathfrak{A} \models \Phi_{g(x)}(W_1/t_1, \dots, W_r/t_r).$$

We shall show that every forcing α-definable set is formally α-definable.

Let var be an effective mapping of the natural numbers onto the variables. Given a natural number x, by X we shall denote the variable $var(x)$.

Let $y_1 < y_2 < \dots < y_k$ be the elements of a finite set D, let Q be one of the quantifiers $\exists$ or $\forall$ and let Φ be an arbitrary formula. Then by $Q(y : y \in D)\Phi$ we shall denote the formula $QY_1 \dots QY_k\Phi$.

Lemma 3.4. *Let $D = \{w_1, \dots, w_r\}$ be a finite and not empty set of natural numbers and x, e be elements of $\mathbb{N}$. Let $\alpha < \omega_1^{CK}$. There exists an uniform effective way to construct a Σ_α^+ formula $\Phi_{D,e,x}^\alpha$ with free variables among $W_1, \dots, W_r$ such that for every finite part δ with $\mathrm{dom}(\delta) = D$ the following equivalence is true:*

$$\mathfrak{A} \models \Phi_{D,e,x}^\alpha(W_1/\delta(w_1), \dots, W_r/\delta(w_r)) \iff \delta \Vdash_\alpha F_e(x).$$

Proof. We shall construct the formula $\Phi_{D,e,x}^\alpha$ by means of effective transfinite recursion on α following the definition of the forcing relation "$\Vdash$".

1) Let $\alpha = 0$. Let $V = \{v : \langle v, x\rangle \in W_e\}$. Consider an element v of V. For every $u \in D_v$ define the atom Π_u as follows:

a) If $u = \langle i, x_1^u, \dots, x_{r_i}^u\rangle$, where $1 \le i \le k$ and all $x_1^u, \dots, x_{r_i}^u$ are elements of D, then let $\Pi_u = T_i(X_1^u, \dots, X_{r_i}^u)$;

b) Let $\Pi_u = W_1 \neq W_1$ in the other cases.

Set $\Pi_v = \bigwedge_{u\in D_v} \Pi_u$ and $\Phi^\alpha_{D,e,x} = \bigvee_{v\in V} \Pi_v$.

2) Let $\alpha = \beta + 1$. Let again $V = \{v : \langle v, x\rangle \in W_e\}$ and $v \in V$. For every $u \in D_v$ define the formula Π_u as follows:

a) If $u = \langle 0, e_u, x_u\rangle$, then let $\Pi_u = \Phi^\beta_{D,e_u,x_u}$;

b) If $u = \langle 1, e_u, x_u\rangle$, then let

$$\Pi_u = \neg[\bigvee_{D^*\supseteq D} (\exists y \in D^* \setminus D)\Phi^\beta_{D^*,e_u,x_u}];$$

c) Let $\Pi_u = \Phi^\beta_{\{0\},0,0} \wedge \neg\Phi^\beta_{\{0\},0,0}$ in the other cases.

Now let $\Pi_v = \bigwedge_{u\in D_v} \Pi_u$ and set $\Phi^\alpha_{D,e,x} = \bigvee_{v\in V} \Pi_v$.

3) Let $\alpha = \lim \alpha(p)$ be a limit ordinal. Let $V = \{v : \langle v, x\rangle \in W_e\}$. Consider a $v \in V$. For every element $u = \langle p_u, e_u, x_u\rangle$ of D_v set $\Pi_u = \Phi^{\alpha(p_u)}_{D,e_u,x_u}$.

Set $\Pi_v = \bigwedge_{u\in D_v} \Pi_u$ and $\Phi^\alpha_{D,e,x} = \bigvee_{v\in V} \Pi_v$.

An easy transfinite induction on α shows that for every α the Σ^+_α formula $\Phi^\alpha_{D,e,x}$ satisfies the requirements of the lemma. $\square$

Theorem 3.4. *Let $\alpha < \omega_1^{CK}$ and let $A \subseteq \mathbb{N}$ be forcing α-definable on $\mathfrak{A}$. Then A is formally α-definable on $\mathfrak{A}$.*

Proof. Suppose that for all $x \in \mathbb{N}$ we have

$$x \in A \iff (\exists\tau \supseteq \delta)(\tau \Vdash_\alpha F_e(x)).$$

Let $D = \mathrm{dom}(\delta) = \{w_1, \dots, w_r\}$ and let $\delta(w_i) = t_i$, $i = 1, \dots, r$. Consider a finite set $D^* \supseteq D$. By the previous lemma

$$\mathfrak{A} \models \exists(y \in D^* \setminus D)\Phi^\alpha_{D^*,e,x}(W_1/t_1, \dots, W_r/t_r)$$

if and only if there exists a finite part τ such that $\mathrm{dom}(\tau) = D^*$, $\tau \supseteq \delta$ and $\tau \Vdash_\alpha F_e(x)$.

Hence we have that for all $x \in \mathbb{N}$ the following equivalence is true:

$$x \in A \iff \mathfrak{A} \models \bigvee_{D^*\supseteq D} \exists(y \in D^* \setminus D)\Phi^\alpha_{D^*,e,x}(W_1/t_1, \dots, W_r/t_r).$$

Set

$$\Phi_{g(x)} = \bigvee_{D^*\supseteq D} \exists(y \in D^* \setminus D)\Phi^\alpha_{D^*,e,x}(W_1, \dots, W_r).$$

Clearly, for all $x \in \mathbb{N}$ we have

$$x \in A \iff \mathfrak{A} \models \Phi_{g(x)}.$$

Hence A is formally α-definable on $\mathfrak{A}$. $\square$

Evidently, every formally α-definable set is α-admissible in all enumerations f of $\mathfrak{A}$. So we have the following theorem:

Theorem 3.5. *Let $A \subseteq \mathbb{N}$ and $\mathbf{a} = d_e(A)$. Let α be a recursive ordinal. Then the following are equivalent:*

(1) $\mathbf{a} \in CS_\alpha(\mathfrak{A})$;

(2) A *is forcing* α*-definable:*

(3) A *is formally* α*-definable:*

(4) A *is* α*-admissible in all enumerations of* $\mathfrak{A}$.

3.3. REPRESENTING THE COUNTABLE IDEALS AS CO-SPECTRA OF STRUCTURES

In this subsection we are going to prove that every countable ideal of enumeration degrees can be represented as a co-spectrum of some structure.

Definition 3.8. Let $\mathfrak{A}$ be a countable structure. The enumeration degree $\mathbf{d}$ is called *co-degree* of $\mathfrak{A}$ if $\mathbf{d}$ is the greatest element of $CS(\mathfrak{A})$. If $\alpha < \omega_1^{CK}$ and $\mathbf{d}$ is the greatest element of CS_α, then $\mathbf{d}$ is called the α-th jump co-degree of $\mathfrak{A}$.

Clearly, if $\mathbf{d}$ is the α-th jump degree of a structure $\mathfrak{A}$, then $\mathbf{d}$ is also the α-th jump co-degree of $\mathfrak{A}$. The vice-versa is not always true. For example, let $\mathfrak{A} = (\mathbb{N}; <, =, \neq)$ be a linear ordering. It is easy to see by a direct analysis of the formally 0-definable on $\mathfrak{A}$ sets that the co-degree of $\mathfrak{A}$ is $\mathbf{0}$. On the other hand, there exist linear orderings without a degree, see [11]. From the results in [8] it follows that the first jump co-degree of $\mathfrak{A}$ is $\mathbf{0}'$ and again there are examples of linear orderings without first jump degree.

Obviously, if a structure $\mathfrak{A}$ has a co-degree, then $CS(\mathfrak{A})$ is a principle ideal. Building on results of Coles, Downey and Slaman [4], we shall show that every principle ideal of enumeration degrees can be represented as $CS(G)$ from some subgroup G of the additive group of the rational numbers $Q = (Q; +, =, \neq)$.

Let us fix a non-trivial group $G \subseteq Q$. Let $a \neq 0$ be an element of G. For every prime number p set

$$h_p(a) = \begin{cases} k, & \text{if } k \text{ is the greatest number such that } p^k | a \text{ in } G, \\ \infty, & \text{if } p^k | a \text{ in } G \text{ for all } k. \end{cases}$$

Let $p_0, p_1, \dots$ be the standard enumeration of the prime numbers and set

$$S_a(G) = \{\langle i, j\rangle : j \leq h_{p_i}(a)\}.$$

It can be easily seen that if a and b are non-zero elements of G, then $S_a(G) \equiv_e S_b(G)$. Let $\mathbf{d}_G = d_e(S_a(G))$, where a is some non-zero element of G.

In [4] it is proved that for every total enumeration degree $\mathbf{a}$ there exists a bijective enumeration f of G such that $f^{-1}(\mathfrak{A}) \in \mathbf{a}$ if and only if $\mathbf{d}_G \leq \mathbf{a}$. Since for every enumeration f we have that $f^{-1}(G)$ is a total set and $\mathbf{d}_G \leq d_e(f^{-1}(G))$, we get the following proposition:

Proposition 3.5. $DS(G) = \{\mathbf{a} : \mathbf{a}$ *is total* & $\mathbf{a} \geq \mathbf{d}_G\}$.

Corollary 3.3. $CS(G) = \{\mathbf{b} : \mathbf{b} \leq \mathbf{d}_G\}$.

Proof. Clearly, $\mathbf{b} \in CS(G)$ if and only if for all total $\mathbf{a} \geq \mathbf{d}_G$, $\mathbf{a} \geq \mathbf{b}$. According Selman's Theorem [14] the last is equivalent to $\mathbf{d}_G \geq \mathbf{b}$. □

Corollary 3.4. *The group* G *has a degree if and only if* $\mathbf{d}_G$ *is total.*

Corollary 3.5. ([4]) *Every group $G \subseteq Q$ has a first jump degree.*

Proof. It is sufficient to show that $\mathbf{d}'_G \in DS_1(G)$. Indeed, by the Jump Inversion Theorem [16] there exists a total degree $\mathbf{a} \geq \mathbf{d}_G$ such that $\mathbf{a}' = \mathbf{d}'_G$. Obviously, $\mathbf{a}' \in DS_1(G)$. □

It remains to see that for every enumeration degree $\mathbf{d}$ there exists a subgroup G of Q such that $\mathbf{d}_G = \mathbf{d}$. Indeed, let $D \subseteq \mathbb{N}$. Consider the set

$$S = \{\langle i, j\rangle : j = 0 \vee j = 1 \ \&\ i \in D\}.$$

It is evident that $S \equiv_e D$. Consider the least subgroup G of Q containing the set $\{1/p_i^j : \langle i, j\rangle \in S\}$. Then $1 \in G$ and $S_1(G) = S$. So, $\mathbf{d}_G = d_e(D)$.

Now let us turn to the representation of an arbitrary countable ideal I of enumeration degrees. Without a loss of generality we may assume that there exists a sequence $\mathbf{b}_0 \leq \mathbf{b}_1 \leq \cdots \leq \mathbf{b}_k \ldots$ of elements of I such that

$$\mathbf{a} \in I \iff (\exists k)(\mathbf{a} \leq \mathbf{b}_k).$$

For every k fix a set $B_k \in \mathbf{b_k}$.

Consider the structure $\mathfrak{A} = (\mathbb{N}; G_\varphi, \sigma, =, \neq)$, where G_φ is the graph of the total recursive function φ such that $\varphi(\langle x, y\rangle) \simeq \langle x+1, y\rangle$ and

$$\sigma = \{\langle x, y\rangle : (\exists k)(y = 2k \vee y = 2k+1 \ \&\ x \in B_k)\}.$$

Proposition 3.6. $CS(\mathfrak{A}) = I$.

Proof. To show that $I \subseteq CS(\mathfrak{A})$, it is sufficient to see that $(\forall k)(\mathbf{b}_k \in CS(\mathfrak{A}))$. Indeed, let us fix a k and let f be an enumeration. Let $f^{-1}(G_\varphi) = G^f$, $f^{-1}(\sigma) = \sigma^f$ and fix a natural number x_k such that $f(x_k) = \langle 0, 2k+1\rangle$. Then for every $x \in \mathbb{N}$ we have

$$x \in B_k \iff (\exists y_1 \ldots \exists y_x)(G^f(x_k, y_1) \ \&\ G^f(y_1, y_2) \ \&\ G^f(y_{x-1}, y_x) \ \&\ \sigma^f(y_x)).$$

Thus $B_k \leq_e f^{-1}(\mathfrak{A})$.

To prove the inverse inclusion, we shall show that if A is a formally definable on $\mathfrak{A}$ set of natural numbers, then $A \leq B_k$ for some k. Let us suppose that g is a recursive function taking values indeces of Σ_0^+ formulae $\Phi_{g(x)}$ with free variables among $W_1, \ldots, W_r$ and $t_1, \ldots, t_k$ are natural numbers such that

$$x \in A \iff \mathfrak{A} \models \Phi_{g(x)}(W_1/t_1, \ldots, W_r/t_r).$$

Without a loss of generality we may assume that every $t_i = \langle 0, l_i\rangle$, where $l_1, \ldots, l_r$ are distinct natural numbers. Assume that $l_1, \ldots, l_s$ are the odd numbers among $l_1, \ldots, l_r$ and let $l_i = 2k_i + 1$, $i = 1, \ldots, s$. Set $k = \max(k_1, \ldots, k_s)$. We shall show that $A \leq_e B_k$. Indeed, let us consider an elementary Σ_0^+ formula

$$S = \exists Y_1 \ldots \exists Y_m M(Y_1, \ldots, Y_m, W_1, \ldots, W_r),$$

where M is a finite conjunction of the atoms $L_1, \ldots, L_p$. We shall show that there exists a uniform recursive procedure, which either decides that $\mathfrak{A} \not\models S(W_1/t_1, \ldots, W_r/t_r)$ or constructs finite sets of natural numbers $E_1, \ldots, E_s$ such that

$$\mathfrak{A} \models S(W_1/t_1, \ldots, W_r/t_r) \iff E_1 \subseteq B_{k_1} \ \&\ \ldots \ \&\ E_s \subseteq B_{k_s}.$$

Substituting all atomic predicates of the form $G_\varphi(Z.T)$ by $T = \varphi(Z)$, we may assume that the predicate G_φ does not occur in S.

1. Check if all L_i are of the form $Z \neq T$ or $\sigma(\varphi^{n_i}(Z))$. If there is an L of the form $Z \neq Z$, then yield $\mathfrak{A} \not\models S(W_1/t_1, \dots, W_r/t_r)$ and go to 6. Otherwise, for $j = 1, \dots s$ set

$$E_j = \{n_i : \sigma(\varphi^{n_i}(W_j)) \in \{L_1, \dots, L_p\}\}$$

and go to 6. If not all L_i are of the form $Z \neq T$ or $\sigma(\varphi^{n_i}(Z))$, then go to 2.

2. Remove all atomic predicates $\varphi^n(W_i) = \varphi^n(W_i)$. If there exists a predicate of the form $\varphi^{n_1}(W_i) = \varphi^{n_2}(W_j)$, where $i \neq j$, then yield $\mathfrak{A} \not\models S(W_1/t_1, \dots, W_r/t_r)$ and go to 6. Otherwise go to 3.

3. Suppose that among $L_1, \dots, L_p$ there exists an atomic predicate L of the form $\varphi^{n_1}(W_j) = \varphi^{n_2}(Z)$, where $n_1 < n_2$. Then $\mathfrak{A} \not\models S(W_1/t_1, \dots, W_r/t_r)$. Go to 6. If no such L exists, go to 4.

4. Suppose that there exists an L which is of the form $\varphi^{n_1}(Z) = \varphi^{n_2}(T)$, where $Z \notin \{W_1, \dots, W_r\}$ and $n_1 \geq n_2$. Remove L from the list and replace in the remaining atomic predicates all occurences of Z by $\varphi^{(n_2 - n_1)}(T)$. Go to 1. Otherwise, check if there exists an L of the form $\varphi^{n_1}(T) = \varphi^{n_2}(Z)$, replace it by $\varphi^{n_1}(Z) = \varphi^{n_2}(T)$ and go to 3. Otherwise go to 5.

5. Consider the first L of the form $\varphi^{n_1}(Z) \neq \varphi^{n_2}(T)$, where $\max(n_1, n_2) > 0$. If the variables Z and T are distinct, then replace it by $Z \neq T$. If $Z = T$, then if $n_1 = n_2$, decide that $\mathfrak{A} \not\models S(W_1/t_1, \dots, W_r/t_r)$ and go to 6. If $n_1 \neq n_2$, then remove L from the list and go to 1. If no such L exists, go to 1.

6. End of the procedure.

Using the above procedure, we may construct an enumeration operator Γ such that for all x

$$\mathfrak{A} \models \Phi_{g(x)}(W_1/t_1, \dots, W_r/t_r) \iff x \in \Gamma(B_k).$$

Thus $A \leq_e B_k$. $\square$

4. PROPERTIES OF THE DEGREE SPECTRA

4.1. GENERAL PROPERTIES OF UPWARDS CLOSED SETS

Definition 4.1. Consider a subset $\mathcal{A}$ of $\mathcal{D}_e$. Say that $\mathcal{A}$ is *upwards closed* if for every $\mathbf{a} \in \mathcal{A}$ all total degrees greater than $\mathbf{a}$ are contained in $\mathcal{A}$.

By Proposition 2.2 every degree spectrum is an upwards closed set of degrees. In this subsection we shall prove some properties of the upwards closed sets of degrees. The next subsection contains specific properties of the degree spectra, i.e. properties which are not true for all upwards closed sets of degrees.

Let $\mathcal{A}$ be an upwards closed set of degrees.

Notice first that if $\mathcal{B} \subseteq \mathcal{A}$, then $co(\mathcal{A}) \subseteq co(\mathcal{B})$.

Proposition 4.1. *Let* $\mathcal{A}_t = \{\mathbf{a} : \mathbf{a} \in \mathcal{A}\ \&\ \mathbf{a}$ *is total*$\}$. *Then* $co(\mathcal{A}) = co(\mathcal{A}_t)$.

Proof. A simple application of Selman's Theorem [14]. Suppose that $\mathbf{b} \in co(\mathcal{A}_t)$. Towards a contradiction assume that $\mathbf{b} \notin co(\mathcal{A})$. Then there exists an element $\mathbf{c} \in \mathcal{A}$ such that $\mathbf{b} \not\leq \mathbf{c}$. By Selman's Theorem there exists a total $\mathbf{a} \geq \mathbf{c}$ such that $\mathbf{b} \not\leq \mathbf{a}$. Clearly, $\mathbf{a} \in \mathcal{A}_t$. A contradiction. □

The next property can be obtained as an application of the Jump Inversion Theorem (JIT) from [17].

Proposition 4.2. *Let* $\mathbf{b}$ *be an arbitrary enumeration degree. Let* α *be a recursive ordinal greater than* 0. *Set*

$$\mathcal{A}_{\mathbf{b},\alpha} = \{\mathbf{a} : \mathbf{a} \in \mathcal{A} \ \&\ \mathbf{b} \leq \mathbf{a}^{(\alpha)}\}.$$

Then $co(\mathcal{A}) = co(\mathcal{A}_{\mathbf{b},\alpha})$.

Proof. Obviously, $co(\mathcal{A}) \subseteq co(\mathcal{A}_{\mathbf{b},\alpha})$. Assume that there exists a degree $\mathbf{c} \in co(\mathcal{A}_{\mathbf{b},\alpha}) \setminus co(\mathcal{A})$. Then there exists an $\mathbf{a} \in \mathcal{A}$ such that $\mathbf{c} \not\leq \mathbf{a}$. By the JIT there exists a total degree $\mathbf{f}$ such that $\mathbf{a} \leq \mathbf{f}$, $\mathbf{b} \leq \mathbf{f}^{(\alpha)}$ and $\mathbf{c} \not\leq \mathbf{f}$. Clearly, $\mathbf{f} \in \mathcal{A}_{\mathbf{b},\alpha}$. A contradiction. □

4.2. SPECIFIC PROPERTIES OF DEGREE SPECTRA

Let us fix an abstract structure $\mathfrak{A}$.

From Proposition 4.2 it follows that the elements of an upwards closed set $\mathcal{A}$ with arbitrary high jumps determine completely the co-set of $\mathcal{A}$. The next theorem shows that the elements of the degree spectrum $DS(\mathfrak{A})$ with low jumps also determine its co-set $CS(\mathfrak{A})$.

Let $\alpha > 0$ be a constructive ordinal and $\mathbf{b} \in DS_\alpha(\mathfrak{A})$. Denote by $\mathcal{A}$ the set $\{\mathbf{a} : \mathbf{a} \in DS(\mathfrak{A}) \ \&\ \mathbf{a}^{(\alpha)} = \mathbf{b}\}$.

Theorem 4.1. $CS(\mathfrak{A}) = co(\mathcal{A})$.

Proof. It is sufficient to show that $co(\mathcal{A}) \subseteq CS(\mathfrak{A})$. Let $\mathbf{c} \in co(\mathcal{A})$ and let C be a set in $\mathbf{c}$. We shall show that C is 0-forcing definable on $\mathfrak{A}$. Evidently, there exists an enumeration g of $\mathfrak{A}$ such that $g^{-1}(\mathfrak{A})^{(\alpha)} \in \mathbf{b}$. Since $\alpha > 0$, $Q = g^{-1}(\mathfrak{A})^{(\alpha)}$ is a total set. Let f be an α, Q-acceptable enumeration. Then $d_e(f^{-1}(\mathfrak{A})) \in \mathcal{A}$ and hence $C \leq_e f^{-1}(\mathfrak{A})$. So C is 0-admissible in all α, Q-acceptable enumerations of $\mathfrak{A}$. By Theorem 3.3, C is 0-forcing definable on $\mathfrak{A}$ and hence $\mathbf{c} \in CS(\mathfrak{A})$. □

There exists upwards closed set of enumeration degrees for which Theorem 4.1 is not true. Indeed, consider two sets of A and B of natural numbers such that $B \not\leq_e A$ and $A \not\leq_e B'$. One may take an arbitrary $B \not\leq_e \emptyset$ and construct the set A as a B'-generic set such that $B \not\leq A$. Let $\mathcal{D} = \{\mathbf{a} : \mathbf{a} \geq d_e(A)\} \cup \{\mathbf{a} : \mathbf{a} \geq d_e(B)\}$. Let $\mathcal{A} = \{\mathbf{a} : \mathbf{a} \in \mathcal{D} \ \&\ \mathbf{a}' = d_e(B)'\}$. Clearly, if $\mathbf{a} \geq d_e(A)$, then $\mathbf{a} \notin \mathcal{A}$. Therefore $d_e(B)$ is the least element of $\mathcal{A}$ and hence $d_e(B) \in co(\mathcal{A})$. On the other hand, $d_e(B) \not\leq d_e(A)$ and hence $d_e(B) \notin co(\mathcal{D})$.

Now we turn to an analog of the Minimal Pair Theorem for the enumeration degrees.

Given a partial mapping f of $\mathbb{N}$ into $\mathbb{N}$, let $f_0 = \lambda x.f(2x)$ and $f_1 = \lambda x.f(2x+1)$.

Definition 4.2. An enumeration f is *splitting* if the functions f_0 and f_1 are enumerations, i.e. f_0 and f_1 are surjective mappings of $\mathbb{N}$ onto $\mathbb{N}$.

Obviously, if f is a splitting enumeration, then both $f_0^{-1}(\mathfrak{A})$ and $f_1^{-1}(\mathfrak{A})$ are enumeration reducible to $f^{-1}(\mathfrak{A})$.

Lemma 4.1. *Let f be an α-generic splitting enumeration of $\mathfrak{A}$. Then both f_0 and f_1 are α-generic enumerations.*

Proof. We shall show that f_0 is α-generic. The proof of the genericity of f_1 is similar. Let $\beta < \alpha$ and let S_0 be an enumeration reducible to $D(\mathfrak{A})^{(\beta)}$ set of finite parts. Denote by S the set $\{\tau : \tau_0 \in S_0\}$. $S \leq_e S_0$ and hence there exists a $\tau \subseteq f$ such that $\tau \in S \vee (\forall \rho \supseteq \tau)(\rho \notin S)$.

Clearly, $\tau_0 \subseteq f_0$ and if $\tau \in S$, then $\tau_0 \in S_0$. Suppose that $(\forall \rho \supseteq \tau)(\rho \notin S)$. Assume that there exists a $\mu \supseteq \tau_0$ such that $\mu \in S_0$. Notice that since $\mu \supseteq \tau_0$, we have that for all x if $\tau(2x) \simeq y$, then $\mu(x) \simeq y$. Let

$$\rho(x) \simeq \begin{cases} \mu(x/2), & \text{if } x \text{ is even,} \\ \tau(x), & \text{if } x \text{ is odd.} \end{cases}$$

Then $\tau \subseteq \rho$ and $\rho_0 = \mu \in S_0$. So, $\rho \in S$. A contradiction. □

Corollary 4.1. *If f is an α-generic splitting enumeration, then $d_e(f_0^{-1}(\mathfrak{A}))^{(\beta)}$ and $d_e(f_1^{-1}(\mathfrak{A}))^{(\beta)}$ do not belong to $CS_\beta(\mathfrak{A})$ for any $\beta < \alpha$.*

Proposition 4.3. *Let f be an α-generic splitting enumeration of $\mathfrak{A}$. Set $\mathbf{f_0} = d_e(f_0^{-1}(\mathfrak{A}))$ and $\mathbf{f_1} = d_e(f_1^{-1}(\mathfrak{A}))$. Then for every β such that $\beta + 1 < \alpha$,*

$$co(\{\mathbf{f_0}^{(\beta)}, \mathbf{f_1}^{(\beta)}\}) = CS_\beta(\mathfrak{A}).$$

Proof. Let $\beta + 1 < \alpha$. It is sufficient to show that if $A \leq_e f_0^{-1}(\mathfrak{A})^{(\beta)}$ and $A \leq_e f_1^{-1}(\mathfrak{A})^{(\beta)}$, then A is β-forcing definable on $\mathfrak{A}$. Indeed, suppose that there exist e_0 and e_1 such that

$$(\forall x)((x \in A \iff f_0 \models_\beta F_{e_0}(x)) \;\&\; (x \in A \iff f_1 \models_\beta F_{e_1}(x))).$$

Consider the set

$$S = \{\tau : (\exists x)(\tau_0 \Vdash_\beta F_{e_0}(x) \;\&\; \tau_1 \Vdash_\beta \neg F_{e_1}(x) \vee \tau_0 \Vdash_\beta \neg F_{e_0}(x) \;\&\; \tau_1 \Vdash_\beta F_{e_1}(x))\}.$$

Clearly, S is an enumeration reducible to $D(\mathfrak{A})^{(\beta+1)}$ and hence there exists a $\tau \subseteq f$ such that $\tau \in S$ or τ has no extensions in S. Assume that $\tau \in S$. Then for some x we have that $f_0 \models_\beta F_{e_0}(x) \;\&\; f_1 \models_\beta \neg F_{e_1}(x)$ or $f_0 \models_\beta \neg F_{e_0}(x) \;\&\; f_1 \models_\beta F_{e_1}(x)$, which is impossible. So, there exists a $\tau \subseteq f$ such that τ has no extensions in S. We shall show that

$$A = \{x : (\exists \rho \supseteq \tau_0)(\rho \Vdash_\beta F_{e_0}(x))\}.$$

Let $x \in A$. Then $f_0 \models_\beta F_{e_0}(x)$ and hence there exists a $\rho \subseteq f_0$ such that $\rho \Vdash_\beta F_{e_0}(x)$. Then $\tau_0 \subseteq f_0$ and hence we may assume that $\tau_0 \subseteq \rho$. Assume now that for some $x \notin A$ there exists a $\rho \supseteq \tau_0$ such that $\rho \Vdash_\beta F_{e_0}(x)$. Then $f_1 \not\models_\beta F_{e_1}(x)$

and hence there exists a $\mu \subseteq f_1$ such that $\mu \Vdash_3 \neg F_{e_1}(x)$. Again we may assume that $\tau_1 \subseteq \mu$. Now let

$$\sigma(x) \simeq \begin{cases} \rho(x/2), & \text{if } x \text{ is even,} \\ \mu(\lfloor x/2 \rfloor), & \text{if } x \text{ is odd.} \end{cases}$$

It is easy to see that $\sigma_0 = \rho$ and $\sigma_1 = \mu$. Therefore $\tau \subseteq \sigma$ and $\sigma \in S$. A contradiction. □

Theorem 4.2. *Let $\alpha < \omega_1^{CK}$ and let $\mathbf{b} \in DS_\alpha(\mathfrak{A})$. There exist elements $\mathbf{f_0}$ and $\mathbf{f_1}$ of $DS(\mathfrak{A})$ such that:*

(1) $\mathbf{f_0}^{(\alpha)} \leq \mathbf{b}$ *and* $\mathbf{f_1}^{(\alpha)} \leq \mathbf{b}$;

(2) *If $\beta < \alpha$, then $\mathbf{f_0}^{(\beta)}$ and $\mathbf{f_1}^{(\beta)}$ do not belong to $CS_\beta(\mathfrak{A})$;*

(3) *If $\beta + 1 < \alpha$, then $co(\{\mathbf{f_0}^{(\beta)}, \mathbf{f_1}^{(\beta)}\}) = CS_\beta(\mathfrak{A})$.*

Proof. Let g be a bijective enumeration of $\mathfrak{A}$ such that $d_e(g^{-1}(\mathfrak{A})^{(\alpha)}) \leq \mathbf{b}$. Denote by $\mathfrak{B}$ the structure $(\mathbb{N}; g^{-1}(R_1), \dots, g^{-1}(R_k))$. Clearly, $D(\mathfrak{B}) \equiv_e g^{-1}(\mathfrak{B})$ and for all β we have that $DS_\beta(\mathfrak{A}) = DS_\beta(\mathfrak{B})$ and $CS_\beta(\mathfrak{A}) = CS_\beta(\mathfrak{B})$. Let f be an α-generic splitting enumeration of $\mathfrak{B}$ such that $f^{-1}(\mathfrak{B})^{(\alpha)} \leq_e D(\mathfrak{B})^{(\alpha)}$. Set $\mathbf{f_0} = d_e(f_0^{-1}(\mathfrak{B}))$ and $\mathbf{f_1} = d_e(f_1^{-1}(\mathfrak{B}))$. Obviously, $\mathbf{f_0}$ and $\mathbf{f_1}$ satisfy the conditions (1) – (3). □

Again we have that Theorem 4.2 is not true for arbitrary upwards closed sets of degrees. Indeed, consider the finite lattice L consisting of the elements $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{a} \wedge \mathbf{b}$, $\mathbf{a} \wedge \mathbf{c}$, $\mathbf{b} \wedge \mathbf{c}$, $\top$, $\bot$ such that $\top$ and $\bot$ are the greatest and the least element of L, respectively, $\mathbf{a} > \mathbf{a} \wedge \mathbf{b}$, $\mathbf{a} > \mathbf{a} \wedge \mathbf{c}$, $\mathbf{b} > \mathbf{a} \wedge \mathbf{b}$, $\mathbf{b} > \mathbf{b} \wedge \mathbf{c}$, $\mathbf{c} > \mathbf{a} \wedge \mathbf{c}$ and $\mathbf{c} > \mathbf{b} \wedge \mathbf{c}$. Since every finite lattice can be embedded in the semilatice of the Turing degrees, see p. 156 of [9], the lattice L can be embedded in $(\mathfrak{D}_T, \leq)$ and hence it can be embedded in $(\mathfrak{D}_e, \leq)$. So we may assume that L is a substructure of $(\mathfrak{D}_e, \leq)$. Let

$$\mathcal{A} = \{\mathbf{d} \in \mathfrak{D}_e : \mathbf{d} \geq \mathbf{a} \vee \mathbf{d} \geq \mathbf{b} \vee \mathbf{d} \geq \mathbf{c}\}.$$

Clearly, $\mathcal{A}$ is an upwards closed set of enumeration degrees. Assume that there exist $\mathbf{f_0}, \mathbf{f_1} \in \mathcal{A}$ such that $co(\{\mathbf{f_0}, \mathbf{f_1}\}) = co(\mathcal{A})$. Let $\mathbf{x_0}, \mathbf{x_1} \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ be such that $\mathbf{f_0} \geq \mathbf{x_0}$ and $\mathbf{f_1} \geq \mathbf{x_1}$. Let $\mathbf{x_2} = \min\{\mathbf{x_0}, \mathbf{x_1}\}$. Then $\mathbf{x_2} \in co(\{\mathbf{f_0}, \mathbf{f_1}\})$, but $\mathbf{x_2} \notin co(\mathcal{A})$. A contradiction.

Now we turn to the third property of $DS(\mathfrak{A})$, showing the existence of enumeration degrees, which are quasi-minimal with respect to $CS(\mathfrak{A})$.

Let $\bot \notin \mathbb{N}$.

Definition 4.3. A *partial finite part* is a finite mapping of $\mathbb{N}$ into $\mathbb{N} \cup \{\bot\}$. A *partial enumeration* is a partial surjective mapping of $\mathbb{N}$ onto $\mathbb{N}$.

From now on, by δ,ρ,τ we shall denote partial finite parts. Given a partial finite part τ and a partial enumeration f, by $\tau \subseteq f$ we shall denote that for all x in dom(τ) either $\tau(x) \simeq \bot$ and $f(x)$ is not defined or $\tau(x) \in \mathbb{N}$ and $f(x) \simeq \tau(x)$.

Let $\mathfrak{A} = (\mathbb{N}; R_1, \ldots, R_k)$ be a structure and f be a partial enumeration. Given a subset A of $\mathbb{N}^a$, let

$$f^{-1}(A) = \{\langle x_1, \ldots, x_a\rangle : x_1, \ldots, x_a \in \mathrm{dom}(f) \ \&\ (f(x_1), \ldots, f(x_a)) \in A\}.$$

Let $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \cdots \oplus f^{-1}(R_k)$. As we shall see later, it could happen that $d_e(f^{-1}(\mathfrak{A})) \notin DS(\mathfrak{A})$. On the other hand, next lemma shows that for every partial enumeration f the enumeration degree of $f^{-1}(\mathfrak{A})$ is "almost" in $DS(\mathfrak{A})$.

Lemma 4.2. *Let X be a total set, let f be a partial enumeration and $f^{-1}(\mathfrak{A}) \leq_e X$. Then $d_e(X) \in DS(\mathfrak{A})$.*

Proof. It is sufficient to show that there exists a total surjective mapping g of $\mathbb{N}$ onto $\mathbb{N}$ such that $g^{-1}(\mathfrak{A}) \leq_e X$. Let $E_f = f^{-1}(" = ")$. Clearly, $E_f \leq_e X$. Since $\mathrm{dom}(f) = \{x : \langle x, x\rangle \in E_f\}$, we get that $\mathrm{dom}(f) \leq_e X$ and hence, since X is a total set, $\mathrm{dom}(f)$ is r.e. in X. Let h be a recursive in X enumeration of $\mathrm{dom}(f)$. Set $g = \lambda n.f(h(n))$. Then for every i, $1 \leq i \leq k$, we have

$$g^{-1}(R_i) = \{\langle n_1, \ldots, n_{r_i}\rangle : \langle h(n_1), \ldots, h(n_{r_i})\rangle \in f^{-1}(R_i)\}.$$

Thus $g^{-1}(\mathfrak{A}) \leq_e X$. □

Corollary 4.2. *For every partial enumeration f the enumeration degree of $f^{-1}(\mathfrak{A})'$ belongs to $DS_1(\mathfrak{A})$.*

Proof. By the Jump Inversion Theorem from [16] there exists a total set F such that $f^{-1}(\mathfrak{A}) \leq_e F$ and $F' \equiv_e f^{-1}(\mathfrak{A})'$. Then $d_e(F) \in DS(\mathfrak{A})$ and, hence, $d_e(F') \in DS_1(\mathfrak{A})$. □

Corollary 4.3. *Let f be a partial enumeration. Then $d_e(f^{-1}(\mathfrak{A}))$ is an upper bound of $CS(\mathfrak{A})$.*

Proof. Let $\mathbf{a} \in CS(\mathfrak{A})$ and let $A \in \mathbf{a}$. Consider a total set X such that $f^{-1}(\mathfrak{A}) \leq_e X$. Then $d_e(X) \in DS(\mathfrak{A})$ and hence $A \leq_e X$. By Selman's Theorem [14], $A \leq_e f^{-1}(\mathfrak{A})$. □

Definition 4.4. Let f be a partial enumeration of $\mathfrak{A}$ and $e, x \in \mathbb{N}$. Then:

(i) $f \models_0 F_e(x)$ iff there exists a v such that $\langle v, x\rangle \in W_e$ and for all $u \in D_v$
$(\exists i)(1 \leq i \leq k \ \&\ u = \langle i, x_1^u, \ldots, x_{r_i}^u\rangle \ \&\ \{x_1^u, \ldots, x_{r_i}^u\} \subseteq \mathrm{dom}(f)\ \&$
$(f(x_1^u), \ldots, f(x_{r_i}^u)) \in R_i)$;

(ii) $f \models_0 \neg F_e(x) \iff f \not\models_0 F_e(x)$.

It is obvious that $A \leq_e f^{-1}(\mathfrak{A})$ iff there exist an e such that

$$(\forall x \in \mathbb{N})(x \in A \iff f \models_0 F_e(x)).$$

Definition 4.5. Let τ be a partial finite part and $e, x \in \mathbb{N}$. Then:

(i) $\tau \Vdash_0 F_e(x)$ iff there exists a v such that $\langle v, x\rangle \in W_e$ and for all $u \in D_v$, $u = \langle i, x_1^u, \ldots, x_{r_i}^u\rangle, 1 \leq i < k$,

$$x_1^u, \ldots, x_{r_i}^u \in \mathrm{dom}(\tau) \ \&\ (\tau(x_1^u), \ldots, \tau(x_{r_i}^u)) \in R_i;$$

(ii) $\tau \Vdash_0 \neg F_e(x) \iff (\forall \rho \supseteq \tau)(\rho \nVdash_0 F_e(x))$.

Definition 4.6. A subset A of $\mathbb{N}$ is *partially forcing definable* on $\mathfrak{A}$ if there exist an $e \in \mathbb{N}$ and a partial finite part δ such that for all natural numbers x,

$$x \in A \iff (\exists \tau \supseteq \delta)(\tau \Vdash_0 F_e(x)).$$

Clearly, if A is partially forcing definable on $\mathfrak{A}$, then $A \leq_e D(\mathfrak{A})$.

Lemma 4.3. *Let $A \subseteq \mathbb{N}$ be partially forcing definable on $\mathfrak{A}$. Then $d_e(A) \in CS(\mathfrak{A})$.*

Proof. Let g be an arbitrary (total) enumeration of $\mathfrak{A}$. Consider a structure $\mathfrak{B}$, which is isomorphic to $\mathfrak{A}$ and such that $D(\mathfrak{B}) \leq_e g^{-1}(\mathfrak{A})$. Then A is partially forcing definable on $\mathfrak{B}$ and hence $A \leq_e D(\mathfrak{B}) \leq_e g^{-1}(\mathfrak{A})$. □

Definition 4.7. A partial enumeration f is *generic* if for every enumeration reducible to $D(\mathfrak{A})$ set S of partial finite parts the following condition holds:

$$(\exists \tau \subseteq f)(\tau \in S \vee (\forall \rho \supseteq \tau)(\rho \notin S)).$$

We shall list some properties of the partial generic enumerations omitting the proofs, since they are similar to the proofs of the respective properties of the total generic enumerations.

Proposition 4.4. (1) *For every partial generic f, $f^{-1}(\mathfrak{A}) \nleq_e D(\mathfrak{A})$. Hence $d_e(f^{-1}(\mathfrak{A})) \notin CS(\mathfrak{A})$.*

(2) *If f is a partial generic enumeration, then*

$$(\forall e, x)(f \models_0 (\neg)F_e(x) \iff (\exists \tau \subseteq f)(\tau \Vdash_0 (\neg)F_e(x))).$$

(3) *There exists a partial generic enumeration $f \leq_e D(\mathfrak{A})'$ such that $f^{-1}(\mathfrak{A}) \leq_e D(\mathfrak{A})'$.*

(4) *If $A \leq_e f^{-1}(\mathfrak{A})$ for all partial generic enumerations f, then A is partially forcing definable on $\mathfrak{A}$.*

Definition 4.8. Given a set $\mathcal{A}$ of enumeration degrees, say that the degree $\mathbf{q}$ is quasi-minimal with respect to $\mathcal{A}$ if the following conditions hold:

(i) $\mathbf{q} \notin co(\mathcal{A})$;
(ii) If $\mathbf{a}$ is a total degree and $\mathbf{a} \geq \mathbf{q}$, then $\mathbf{a} \in \mathcal{A}$;
(iii) If $\mathbf{a}$ is a total degree and $\mathbf{a} \leq \mathbf{q}$, then $\mathbf{a} \in co(\mathcal{A})$.

Notice that from (ii) it follows by Selman's Theorem that every quasi-minimal degree is an upper bound of $co(\mathcal{A})$.

If for some $\mathbf{d} \in \mathfrak{D}_e$, $\mathcal{A} = \{\mathbf{a} : \mathbf{a} \geq \mathbf{d}\}$, then a degree is quasi-minimal with respect to $\mathcal{A}$ iff it is quasi-minimal over $\mathbf{d}$.

Theorem 4.3. *Let f be a partial generic enumeration of $\mathfrak{A}$. Then $d_e(f^{-1}(\mathfrak{A}))$ is quasi-minimal with respect to $DS(\mathfrak{A})$.*

Proof. It is sufficient to show that if ψ is a total function and $\psi \leq_e f^{-1}(\mathfrak{A})$, then $d_e(\psi) \in CS(\mathfrak{A})$. Suppose that ψ is a total function and

$$(\forall x, y \in \mathbb{N})(\psi(x) \simeq y \iff f \models_0 F_e(\langle x, y\rangle)).$$

Consider the set

$$S_0 = \{\rho : (\exists x, y_1 \neq y_2)(\rho \Vdash_0 F_e(\langle x, y_1\rangle) \ \&\ \rho \Vdash_0 F_e(\langle x, y_2\rangle))\}.$$

Since $S_0 \leq_e D(\mathfrak{A})$, we have that there exists a partial finite part $\tau_0 \subseteq f$ such that either $\tau_0 \in S_0$ or $(\forall \rho \supseteq \tau_0)(\rho \notin S_0)$. Assume that $\tau_0 \in S_0$. Then there exist $x, y_1 \neq y_2$ such that $f \models_0 F_e(\langle x, y_1\rangle)$ and $f \models_0 F_e(\langle x, y_2\rangle)$. Then $\psi(x) \simeq y_1$ and $\psi(x) \simeq y_2$, which is impossible. So, $(\forall \rho \supseteq \tau_0)(\rho \notin S_0)$.

Let

$$\begin{aligned} S_1 = \{\rho :& (\exists \tau \supseteq \tau_0)(\exists \delta_1 \supseteq \tau)(\exists \delta_2 \supseteq \tau)(\exists x, y_1 \neq y_2)(\tau \subseteq \rho \ \&\ \delta_1 \Vdash_0 F_e(\langle x, y_1\rangle) \ \&\\ & \delta_2 \Vdash_0 F_e(\langle x, y_2\rangle) \ \&\ \mathrm{dom}(\rho) = \mathrm{dom}(\delta_1) \cup \mathrm{dom}(\delta_2) \ \&\\ & (\forall x \in \mathrm{dom}(\rho) \setminus \mathrm{dom}(\tau))(\rho(x) \simeq \bot))\}. \end{aligned}$$

Again we have $S_1 \leq_e D(\mathfrak{A})$ and hence there exists a $\tau_1 \subseteq f$ such that either $\tau_1 \in S_1$ or $(\forall \rho \supseteq \tau_1)(\rho \notin S_1)$.

Assume $\tau_1 \in S_1$. Then there exists a τ such that $\tau_0 \subseteq \tau \subseteq \tau_1$ and for some $\delta_1 \supseteq \tau$, $\delta_2 \supseteq \tau$ and $x, y_1 \neq y_2 \in \mathbb{N}$ we have

$$\begin{aligned} &\delta_1 \Vdash_0 F_e(\langle x, y_1\rangle) \ \&\ \delta_2 \Vdash_0 F_e(\langle x, y_2\rangle) \ \&\ \mathrm{dom}(\tau_1) = \mathrm{dom}(\delta_1) \cup \mathrm{dom}(\delta_2) \ \&\\ &(\forall x \in \mathrm{dom}(\tau_1) \setminus \mathrm{dom}(\tau))(\tau_1(x) \simeq \bot). \end{aligned}$$

Let $\psi(x) \simeq y$. Then $f \models_0 F_e(\langle x, y\rangle)$. Hence there exists a $\rho \supseteq \tau_1$ such that $\rho \Vdash_0 F_e(\langle x, y\rangle)$. Let $y \neq y_1$. Define the partial finite part ρ_0 as follows:

$$\rho_0(x) \simeq \begin{cases} \delta_1(x), & \text{if } x \in \mathrm{dom}(\delta_1),\\ \rho(x), & \text{if } x \in \mathrm{dom}(\rho) \setminus \mathrm{dom}(\delta_1). \end{cases}$$

Then $\tau_0 \subseteq \rho_0$, $\delta_1 \subseteq \rho_0$ and for all $x \in \mathrm{dom}(\rho)$ if $\rho(x) \not\simeq \bot$, then $\rho(x) \simeq \rho_0(x)$. Hence $\rho_0 \Vdash_0 F_e(\langle x, y_1\rangle)$ and $\rho_0 \Vdash_0 F_e(\langle x, y\rangle)$. So, $\rho_0 \in S_0$. A contradiction.

Thus, if $\rho \supseteq \tau_1$, then $\rho \notin S_1$.

Let $\tau = \tau_1 \cup \tau_0$. Notice that $\tau \subseteq f$. We shall show that

$$\psi(x) \simeq y \iff (\exists \delta \supseteq \tau)(\delta \Vdash_0 F_e(\langle x, y\rangle)).$$

The left to right implication is trivial. Assume that $\delta_1 \supseteq \tau$, $\delta_1 \Vdash_0 F_e(\langle x, y_1\rangle)$, $\psi(x) \simeq y_2$ and $y_1 \neq y_2$. Then there exists a $\delta_2 \supseteq \tau$ such that $\delta_2 \Vdash_0 F_e(\langle x, y_2\rangle)$. Set

$$\rho(x) \simeq \begin{cases} \tau(x), & \text{if } x \in \mathrm{dom}(\tau),\\ \bot, & \text{if } x \in (\mathrm{dom}(\delta_1) \cup \mathrm{dom}(\delta_2)) \setminus \mathrm{dom}(\tau). \end{cases}$$

Then $\rho \supseteq \tau_1$ and $\rho \in S_1$. A contradiction.

Thus ψ is partially forcing definable and hence $d_e(\psi) \in CS(\mathfrak{A})$. □

As we have already pointed out, not every structure has a degree, i.e. it is not generally true that the set $DS(\mathfrak{A})$ has a least element. The next theorem shows that if $\mathfrak{A}$ has no degree, then for every countable subset $\mathcal{B} \subseteq DS(\mathfrak{A})$ of total enumeration degrees there exists an element $\mathbf{a}$ of $DS(\mathfrak{A})$ such that $(\forall \mathbf{b} \in \mathcal{B})(\mathbf{b} \not\leq \mathbf{a})$.

Definition 4.9. Let $\mathcal{A}$ be a set of enumeration degree. The subset $\mathcal{B}$ of $\mathcal{A}$ is called *base* of $\mathcal{A}$ if for every element $\mathbf{a}$ of $\mathcal{A}$ there exists an element $\mathbf{b} \in \mathcal{B}$ such that $\mathbf{b} \leq \mathbf{a}$.

We need the following lemma, which can be proved by a minor modification of the proof of Selman's Theorem presented in [16]:

Lemma 4.4. *Let $Q \subseteq \mathbb{N}$ and let $\{B_n\}_{n\in\omega}$ be a sequence of sets of natural numbers such that $(\forall n)(B_n \not\leq_e Q)$. Then there exists a total set F such that $Q \leq_e F$ and $(\forall n)(B_n \not\leq_e F)$.*

Theorem 4.4. *Let $\mathcal{A}$ be a set of enumeration degrees possessing a quasi-minimal degree $\mathbf{q}$. Suppose that there exists a countable base $\mathcal{B}$ of $\mathcal{A}$ consisting of total degrees. Then $\mathcal{A}$ has a least element.*

Proof. Towards a contradiction assume that for every $\mathbf{b} \in \mathcal{B}$ we have $\mathbf{b} \not\leq \mathbf{q}$. Let $Q \in \mathbf{q}$ and $\{B_n : n \in \omega\}$ be a sequence of sets such that $\mathcal{B} = \{d_e(B_n) : n \in \omega\}$. Clearly, for all n, $B_n \not\leq_e Q$. Let F be a total set such that $Q \leq_e F$ and $(\forall n)(B_n \not\leq_e F)$. Set $\mathbf{f} = d_e(F)$. Then $\mathbf{f}$ is in $\mathcal{A}$ and for every $\mathbf{b} \in \mathcal{B}$ we have $\mathbf{b} \not\leq \mathbf{f}$. A contradiction. So there exists a $\mathbf{b} \in \mathcal{B}$ such that $\mathbf{b} \leq \mathbf{q}$. Since $\mathbf{b}$ is a total degree, $\mathbf{b} \in co(\mathcal{A})$. Therefore $\mathbf{b}$ is the least element of $\mathcal{A}$. □

Corollary 4.4. *If $DS(\mathfrak{A})$ has a countable base of total enumeration degrees, then $DS(\mathfrak{A})$ has a least element.*

Now it is easy to construct an upwards closed set $\mathcal{A}$ of degrees, which does not possess a quasi-minimal degree. Indeed, let $\mathbf{a}$ and $\mathbf{b}$ be two incomparable total degrees. Let $\mathcal{A} = \{\mathbf{c} : \mathbf{c} \geq \mathbf{a} \vee \mathbf{c} \geq \mathbf{b}\}$. Then $\mathcal{A}$ has a countable base of total degrees, but it has not a least element. So, $\mathcal{A}$ has no quasi-minimal degree.

Corollary 4.4 remains true if we consider the more restrictive definition of $DS(\mathfrak{A})$, which takes into account only the bijective enumerations of $\mathfrak{A}$. Let

$$\overline{DS}(\mathfrak{A}) = \{d_e(f^{-1}(\mathfrak{A})) : f \text{ is a bijective enumeration of } \mathfrak{A}\}.$$

Corollary 4.5. *Let $\overline{DS}(\mathfrak{A})$ have a countable base $\mathcal{B}$. Then $\overline{DS}(\mathfrak{A})$ has a least element.*

Proof. According Proposition 2.1, if $DS(\mathfrak{A})$ has a least element $\mathbf{b}$, then $\mathbf{b}$ will be the least element of $\overline{DS}(\mathfrak{A})$. So, it is sufficient to show that $DS(\mathfrak{A})$ has a least element. We shall show that $\mathcal{B}$ is a base of $DS(\mathfrak{A})$. Indeed, consider an element $\mathbf{a}$ of $DS(\mathfrak{A})$. By Proposition 2.1, there exists a $\mathbf{d} \in \overline{DS}(\mathfrak{A})$ such that $\mathbf{d} \leq \mathbf{a}$. Clearly, there exists a $\mathbf{b} \in \mathcal{B}$ such that $\mathbf{b} \leq \mathbf{d} \leq \mathbf{a}$. □

Finally, we would like to point out the following application of the existence of a quasi-minimal with respect to $DS(\mathfrak{A})$ degree.

Definition 4.10. The structure $\mathfrak{B}$ is called quasi-minimal with respect to $\mathfrak{A}$ if the following are true:

(i) $DS(\mathfrak{B}) \subseteq DS(\mathfrak{A})$;

(ii) $CS(\mathfrak{A}) \neq CS(\mathfrak{B})$;

(iii) If $\mathbf{a}$ is a total degree in $CS(\mathfrak{B})$, then $\mathbf{a} \in CS(\mathfrak{A})$.

Theorem 4.5. *There exists a quasi-minimal with respect to $\mathfrak{A}$ structure.*

Proof. Let $\mathbf{q}$ be a quasi-minimal with respect to $DS(\mathfrak{A})$ degree. Consider a subgroup G of the group of the rational numbers such that

$$DS(G) = \{\mathbf{a} : \mathbf{a} \text{ is total and } \mathbf{q} \leq \mathbf{a}\}.$$

Now (i) is obvious, (ii) follows from the fact that $\mathbf{q} \in CS(G)$, but $\mathbf{q} \notin CS(\mathfrak{A})$, and (iii) follows from the quasi-minimality of $\mathbf{q}$. □

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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GENERALIZED TURÁN'S GRAPH THEOREM

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Let G be an n-vertex graph and there is a vertex of G which is contained in maximum number of p-cliques, but is not contained in $(s+1)$-clique, where $2 \leq p \leq \min(s, n)$. Then the number of p-cliques of G is less than the number of p-cliques in the n-vertex S-partite Turán's graph $T_s(n)$ or $G = T_s(n)$.

Keywords: complete s-partite graph, Turán's graph
2000 MSC: *05C35*

One of the fundamental results in graph theory is the theorem of P. Turán, proved in 1941, [5]. It generalizes a result of Mantel from 1906, [4], saying that if a graph on n vertices has more than $n^2/4$ edges, then this graph necessarily contains a triangle.

Turán's theorem was significantly generalized by Zykov in 1949, [6]. This generalization, unlike Turán's theorem, is not so popular. In this article we present a method to prove Zykov's theorem and its extension, used by us for solving similar problems (see [1], [2] and [3]). Let us fix some notations. We consider graphs $G = (V, E)$, where V is the set of vertices and $E \subseteq \binom{V}{2}$ is the set of edges. If $\{u, v\} \in E$, we say that the vertices u and v are adjacent. We call a *p-clique* of G a set of p vertices, each two of which are adjacent. The number of p-cliques of the graph G will be denoted by $c_p(G)$, and the number of p-cliques containing a vertex v by $c_p(v)$.

Let $G_1 = (V_1, E_1), G_2 = (V_2, E_2), \dots, G_s = (V_s, E_s)$ be graphs such that $V_i \cap V_j = \emptyset$, $i \neq j$. We denote by $G_1 + G_2 + \cdots + G_s$ the graph $G = (V, E)$ with

$$V = V_1 \cup V_2 \cup \cdots \cup V_s \quad \text{and} \quad E = E_1 \cup E_2 \cup \cdots \cup E_s \cup E',$$

where E' consists of all 2-element subsets $\{u, v\}$, $u \in V_i$, $v \in V_j$, $i \neq j$.

Consider a graph with n vertices. If each two of them are adjacent, we denote this graph by K_n, and if no two are adjacent – by $\overline{K}_n$. The graph $\overline{K}_{n_1} + \cdots + \overline{K}_{n_s}$ will be denoted by $K(n_1, \ldots, n_s)$. Obviously, $K(n_1, \ldots, n_s)$ is a complete s-partite graph. If $n_1 + \cdots + n_s = n$ and $|n_i - n_j| \leq 1$ for all i, j, then $K(n_1, \ldots, n_s)$ is denoted by $T_s(n)$ and is called s-partite n-vertex Turán's graph. Clearly, $T_s(n) = K_n$ for $s \geq n$.

Turán's theorem. ([5]) *Let s and n be positive integers and G be an n-vertex graph without $(s+1)$-cliques. Then*

$$c_2(G) \leq c_2\left(T_s(n)\right)$$

and $c_2(G) = c_2\left(T_s(n)\right)$ only if $G = T_s(n)$.

Zykov's theorem. ([6]) *Let p, s and n be positive integers and G be an n-vertex graph without $(s+1)$-cliques. Then:*

(a) $c_p(G) \leq c_p\left(T_s(n)\right)$;

(b) *if $c_p(G) = c_p\left(T_s(n)\right)$ and $2 \leq p \leq \min(n, s)$, then $G = T_s(n)$.*

A special case of Zykov's theorem is the following

Lemma. *Let p, s and n be positive integers and $2 \leq p \leq \min(n, s)$. Then*

$$c_p\Big(K(n_1, n_2, \ldots, n_s)\Big) \leq c_p(T_s(n))$$

for each s-tuple $(n_1, n_2, \ldots, n_s)$ of nonnegative integers n_i such that $n_1 + n_2 + \cdots + n_s = n$. The equality is possible only if $K(n_1, n_2, \ldots, n_s) = T_s(n)$.

Proof. Suppose that $n_1, n_2, \ldots, n_s$ are such that $c_p\Big(K(n_1, n_2, \ldots, n_s)\Big)$ is maximal. Let also $n_1 = \max\{n_1, n_2, \ldots, n_s\}$ and $n_2 = \min\{n_1, n_2, \ldots, n_s\}$.

For $2 \leq p \leq \min(s, n)$ we have

$$\begin{aligned} c_p\Big(K(n_1, n_2, \ldots, n_s)\Big) &= \sum\{n_{i_1} \ldots n_{i_p} \mid 1 \leq i_1 < i_2 < \cdots < i_p \leq s\} \\ &= n_1 n_2 M + (n_1 + n_2)N + P, \end{aligned}$$

where M, N and P do not depend on n_1 and n_2 and $M > 0$. Hence

$$c_p\Big(K(n_1 - 1, n_2 + 1, n_3, \ldots, n_s)\Big) - c_p\Big(K(n_1, n_2, \ldots, n_s)\Big) = M(n_1 - n_2 - 1).$$

The maximality of $c_p\Big(K(n_1, n_2, \ldots, n_s)\Big)$ implies $n_1 - n_2 \leq 1$. From this inequality it follows $K(n_1, n_2, \ldots, n_s) = T_s(n)$.

Proof of Zykov's theorem. Let v_0 be a vertex of the graph G which is contained in a maximum number of p-cliques, i. e. $c_p(v) \leq c_p(v_0)$ for each vertex v. Denote by A the set of vertices v of G, $v \neq v_0$, such that both v and v_0 are contained in some p-clique of the graph G, and by B the set of the remaining vertices of G. Let $\langle A \rangle$ be the subgraph of G generated by A (the vertex set of $\langle A \rangle$ is A and two vertices are adjacent in $\langle A \rangle$ if and only if they are adjacent in G).

Each p-clique of G is either entirely contained in A or has at least one vertex in B. Hence

$$c_p(G) \leq c_p(\langle A \rangle) + \sum_{v \in B} c_p(v), \tag{1}$$

with equality if and only if each p-clique of G has at most one vertex in B. Obviously, $c_p(v_0) = c_{p-1}(\langle A \rangle)$ for $p \geq 2$, and since $c_p(v) \leq c_p(v_0)$ for each vertex v,

$$c_p(v) \leq c_{p-1}(\langle A \rangle) \quad \text{for each vertex } v \text{ in } B \text{ and } p \geq 2. \tag{2}$$

If $k = |A|$ and $p \geq 2$, it follows from (1) and (2) that

$$c_p(G) \leq c_p(\langle A \rangle) + (n-k)c_{p-1}(\langle A \rangle). \tag{3}$$

Equality holds in (3) if and only if it holds in (1) and (2), that is, when there are no p-cliques with more than one vertex in B, and each vertex of B is adjacent to the vertices of each $(p-1)$-clique of $\langle A \rangle$. In the special case $p = s = 2$, equality occurs in (3) if and only if $G = K(k, n-k)$.

We prove the inequality (a) by induction on s. The base $s = 1$ is clear, since in this case $G = \overline{K}_n$.

For the inductive step, assume that $s \geq 2$. Suppose first that $p = 1$. Then $c_1(G) = c_1(T_s(n)) = n$. Let $p \geq 2$. If $c_p(v_0) = 0$, then $c_p(G) = 0$ and (a) is obvious. Let $c_p(v_0) > 0$, i. e. $A \neq \emptyset$. Note that $\langle A \rangle$ does not contain s-cliques, since G does not contain $(s+1)$-cliques. Applying the inductive hypothesis for $\langle A \rangle$, we conclude that if $|A| = k$, then

$$c_p(\langle A \rangle) \leq c_p(T_{s-1}(k)), \tag{4}$$

$$c_{p-1}(\langle A \rangle) \leq c_{p-1}(T_{s-1}(k)). \tag{5}$$

It follows from (3) – (5) that

$$c_p(G) \leq c_p(T_{s-1}(k)) + (n-k)c_{p-1}(T_{s-1}(k)). \tag{6}$$

Set $\Gamma = \overline{K}_{n-k} + T_{s-1}(k)$. Clearly,

$$c_p(\Gamma) = c_p(T_{s-1}(k)) + (n-k)c_{p-1}(T_{s-1}(k)). \tag{7}$$

The lemma, applied to the graph Γ, yields

$$c_p(\Gamma) \leq c_p(T_s(n)). \tag{8}$$

The inequality (a) now follows from (6) – (8).

Passing on to (b), let G be a graph with n vertices without $(s+1)$-cliques, $2 \le p \le \min(s,n)$ and

$$c_p(G) = c_p(T_s(n)). \tag{9}$$

We prove the equality $G = T_s(n)$ by induction on s. Note first that the equality (9) implies equalities in (3) – (6) and (8). By the assumption $2 \le p \le \min(s,n)$, the minimal admissible value of s is 2.

The base of the induction is then $s = 2$; in this case $p = 2$. Let G be a graph with n vertices without 3-cliques satisfying (9) for $p = 2$. Then there is equality in (3) and, as pointed out above, $G = K(k, n-k)$. In view of this, $c_2(K(k, n-k)) = c_2(T_s(n))$. The lemma implies $K(k, n-k) = T_s(n)$ and so $G = T_s(n)$.

Assume now $s \ge 3$ and that (b) holds for graphs without s-cliques. We start the inductive step by noting that $k \ge p-1$. Indeed, it follows from $p \le \min(n,s)$ that $c_p(T_s(n)) > 0$, and (9) implies $c_p(G) > 0$. Thus $c_p(v_0) = c_{p-1}(\langle A \rangle) > 0$, which clearly yields $k = |A| \ge p-1$.

Now we prove that

$$\langle A \rangle = T_{s-1}(k). \tag{10}$$

The cases $p \ge 3$ and $p = 2$ will be treated separately. Let $p \ge 3$. Then $2 \le p-1$. Also, $p-1 \le \min(s-1, k)$. By the inductive hypothesis the equality in (5) implies (10). We are left with the case $p = 2$. If $k \ge 2$, then $p = 2 \le \min(s-1, k)$. So, by the inductive hypothesis, the equality in (4) implies (10). If $k = 1$, (10) holds trivially, because $\langle A \rangle = T_{s-1}(1) = K_1$.

Based on (10), we prove that $G = \Gamma$. It follows from $p-1 \le \min(s-1, k)$ that each vertex of $T_{s-1}(k)$ is a vertex of a $(p-1)$-clique. Since there is equality in (3), we conclude that each vertex of A is adjacent to each vertex of B. On the other hand, B does not contain adjacent vertices. Otherwise, two such vertices, together with $(p-2)$-clique of A, would form a p-clique containing two vertices of B, contradicting the fact that there is equality in (3). It follows from this argument and (10) that $G = \Gamma$.

By the lemma the equality in (8) yields $\Gamma = T_s(n)$, and so $G = T_s(n)$.

The proof of Zykov's theorem is complete. Instead of $c_{s+1}(G) = 0$ we have used the weaker condition $c_{s+1}(v_0) = 0$. Hence, this proof, actually, establishes the following stronger statement:

Theorem. *Let p, s and n be positive integers and $2 \le p \le \min(s,n)$. Let v_0 be a vertex of an n-vertex graph G such that $c_p(v_0) = \max\{c_p(v) \mid v \in G\}$ and v_0 is not contained in an $(s+1)$-clique. Then the inequality (a) and the statement (b) of the theorem of Zykov hold.*

In conclusion, let us note that a direct counting argument for the p-cliques of $T_s(n)$ gives

$$c_p(T_s(n)) = \sum_{t=0}^{p} \binom{\nu}{t} \binom{s-t}{p-t} k^{p-t},$$

where $n = ks + \nu$, $0 \leq \nu < s$.

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
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BOUNDS ON THE VERTEX FOLKMAN NUMBER $F(4,4;5)$

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For a graph G the symbol $G \to (4,4)$ means that in every 2-coloring of the vertices of G there exists a monochromatic K_4. For the vertex Folkman number

$$F(4,4;5) = \min\{|V(G)| : G \to (4,4) \text{ and } K_5 \not\subset G\}$$

we show that $16 \leqq F(4,4;5) \leqq 35$.

Keywords: Folkman numbers, Folkman graphs

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1. NOTATION

We consider only finite, non-oriented graphs, without loops and multiple edges. We call a p-clique of the graph G a set of p vertices, each two of which are adjacent. The largest positive integer p such that the graph G contains a p-clique is denoted by $\mathrm{cl}(G)$.

In this paper we shall use also the following notation:

$V(G)$ — the vertex set of the graph G;

$E(G)$ — the edge set of the graph G;

$\overline{G}$ — the complement of G;

$G[X]$, $X \subseteq V(G)$ — the subgraph of G induced by X;

$G - X$, $X \subseteq V(G)$ — the subgraph of G induced by $V(G) \setminus X$;

$N_G(v)$, $v \in V(G)$ — the set of all vertices of G adjacent to v in G;

K_n — a complete graph on n vertices;

C_n — a simple cycle on n vertices;

$\alpha(G)$ — a vertex independence number of G, i.e. $\alpha(G) = \mathrm{cl}(\overline{G})$.

Let G_1 and G_2 be two graphs without common vertices. We denote by $G_1 + G_2$ the graph G for which $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup E'$, where $E' = \{[x, y] : x \in V(G_1),\ y \in V(G_2)\}$.

Let $G_1, \ldots, G_k$ be graphs, $V(G_i) \cap V(G_j) = \emptyset$, $i \neq j$. We denote by $\bigcup_{i=1}^{k} G_i$ the graph G for which $V(G) = \bigcup_{i=1}^{k} V(G_i)$, $E(G) = \bigcup_{i=1}^{k} E(G_i)$.

The Ramsey number $R(p, q)$ is the smallest natural number n such that for an arbitrary n-vertex graph G either $\mathrm{cl}(G) \geqq p$ or $\alpha(G) \geqq q$.

2. VERTEX FOLKMAN NUMBERS

Definition 2.1. Let G be a graph and p, q be natural numbers. A 2-coloring

$$V(G) = V_1 \cup V_2, \quad V_1 \cap V_2 = \emptyset$$

of the vertices of G is said to be (p, q)-free if V_1 contains no p-cliques and V_2 contains no q-cliques of G. The symbol $G \to (p, q)$ means that every 2-coloring of $V(G)$ is not (p, q)-free.

Define

$$F(p, q; s) = \min\{|V(G)| : G \to (p, q) \text{ and } \mathrm{cl}(G) < s\}.$$

Clearly, $G \to (p, q) \ \Rightarrow\ \mathrm{cl}(G) \geqq \max\{p, q\}$. Folkman [1] has proved that there exists a graph G such that $G \to (p, q)$ and $\mathrm{cl}(G) = \max\{p, q\}$. Therefore

$$F(p, q; s) \text{ exist} \iff s > \max\{p, q\} \tag{1}$$

and they are called vertex Folkman numbers.

Obviously, $K_{p+q-1} \to (p, q)$ and $K_{p+q-2} \nrightarrow (p, q)$. Hence

$$F(p, q; s) = p + q - 1, \text{ if } s > p + q - 1. \tag{2}$$

By (1), the numbers $F(p, q; p + q - 1)$ exist only if $p + q - 1 \geqq \max\{p, q\} + 1$. For these numbers the following result is known ([3]):

$$F(p, q; p + q - 1) = p + q - 1 + \max\{p, q\}. \tag{3}$$

For the numbers $F(p, p; p + 1)$ in [4] it has been shown that

$$3p - 2 \leqq F(p, p; p + 1) \leqq \lfloor 2p!\,(e - 1) \rfloor - 1. \tag{4}$$

In [7] it has been proved that

$$F(p, p; p + 1) \leqq \lfloor p!\, e \rfloor - 2, \quad p \geqq 3. \tag{5}$$

For multicoloring vertex Folkman numbers see [9].

By (1), the numbers $F(3,3;s)$ exist only if $s \geqq 4$. For these numbers it is known that

$$F(3,3;s) = \begin{cases} 5, & \text{if } s \geqq 6, \text{ according to (2);} \\ 8, & \text{if } s = 5, \text{ according to (3);} \\ 14, & \text{if } s = 4. \end{cases} \tag{6}$$

The inequality $F(3,3;4) \leqq 14$ is proved in [6] and the opposite inequality $F(3,3;4) \geqq 14$ is verified by means of computer in [10].

By (1), the numbers $F(4,4;s)$ exist only if $s \geqq 5$. It is known that

$$F(4,4;s) = \begin{cases} 7, & \text{if } s \geqq 8, \text{ according to (2);} \\ 11, & \text{if } s = 7, \text{ according to (3);} \\ 14, & \text{if } s = 6. \end{cases} \tag{7}$$

The inequality $F(4,4;6) \leqq 14$ is proved in [8] and the inequality $F(4,4;6) \geqq 14$ is proved in [5]. By (4), we have $10 \leqq F(4,4;5) \leqq 81$. From (5) it follows that $F(4,4;5) \leqq 63$.

Our main result is the following

Theorem. $16 \leqq F(4,4;5) \leqq 35$.

4. PROOF OF THE INEQUALITY $F(4,4;5) \leqq 61$

Let $V(C_7) = \{v_1, \dots, v_7\}$ and $E(C_7) = \{[v_i, v_{i+1}],\ i = 1, \dots, 6\} \cup \{[v_1, v_7]\}$. Consider the set $V_1 = \{v_2, v_3, v_6, v_7\} \subseteq V(C_7)$. Define $V_i = \sigma^{i-1}(V_1)$, $i = 1, \dots, 7$, where $\sigma(v_i) = v_{i+1}$, $i = 1, \dots, 6$, and $\sigma(v_7) = v_1$. We denote by Γ the extension of $\overline{C}_7$, constructed by adding the new vertices $u_1, \dots, u_7$, each two of which are not adjacent and such that $N_\Gamma(u_i) = V_i$, $i = 1, \dots, 7$. The graph Γ_i (Fig. 1) is a copy of Γ such that the map $v_k \to v_k^i$, $u_k \to u_k^i$, $k = 1, \dots, 7$, is an isomorphism between Γ and Γ_i.

Proposition 4.1. ([6]) $\Gamma_i \to (3,3)$ *and* $\mathrm{cl}(\Gamma_i) = 3$.

Proposition 4.2. *Let G be a graph such that $G \to (p,p)$. Let $V_1 \cup V_2$ be a $(p+1, p+1)$-free 2-coloring of the vertices of $\overline{K}_2 + G$, where $V(\overline{K}_2) = \{u, v\}$. Then $u, v \in V_1$ or $u, v \in V_2$.*

Proof. Assume the opposite, i.e. $u \in V_1$ and $v \in V_2$. Then $(V_1 \setminus u) \cup (V_2 \setminus v)$ is a (p,p)-free coloring of $V(G)$, which is a contradiction. □

Let G be a graph. The graph $K_1 + G$, where $V(K_1) = v$, is given on Fig. 2.

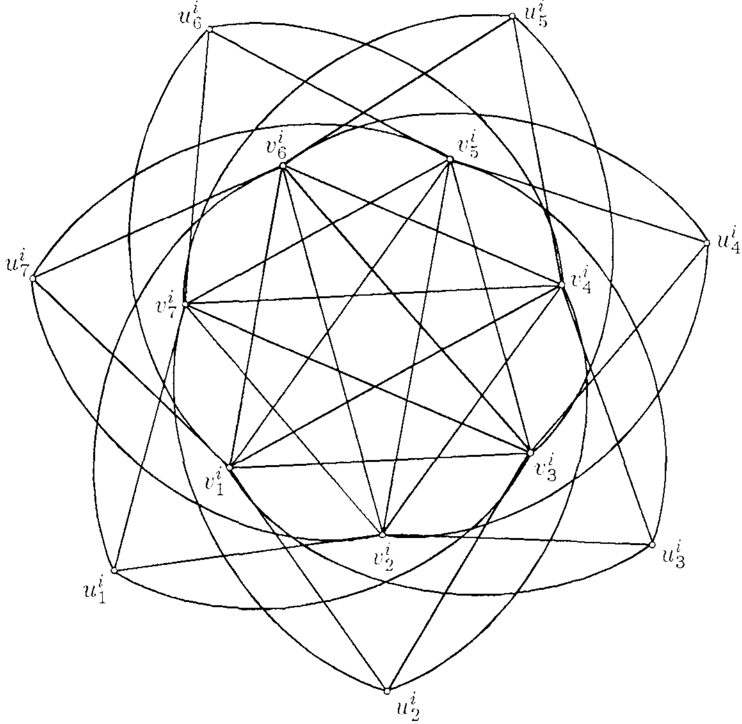

Fig. 1. The graphs Γ_i, $i = 1, 2, 3, 4$

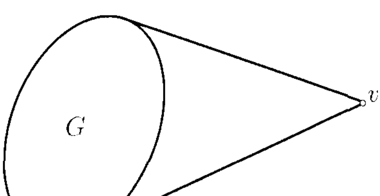

Fig. 2. The graph $K_1 + G = \{v\} + G$

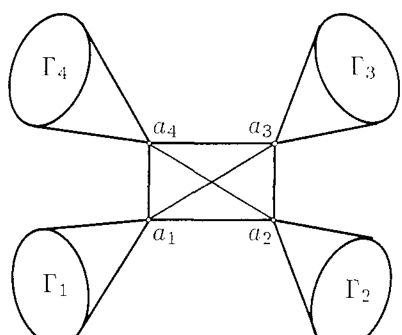

Fig. 3. The graph P

Consider the graph P with 60 vertices shown in Fig. 3, where Γ_1, Γ_2, Γ_3, Γ_4 are given in Fig. 1. We denote by Q the extension of P, constructed by adding the new vertex b such that $N_Q(b) = \bigcup_{i=1}^{4} V(\Gamma_i)$.

Proposition 4.3. *$Q \rightarrow (4, 4)$ and* $\mathrm{cl}(Q) = 4$.

Proof. Since $\mathrm{cl}(\Gamma_i) = 3$, $i = 1, 2, 3, 4$ (Proposition 4.1), we have $\mathrm{cl}(Q) = 4$. Assume that $Q \nrightarrow (4, 4)$ and let $V_1 \cup V_2$ be a $(4, 4)$-free 2-coloring of $V(Q)$. Without a loss of generality, we can assume that $b \in V_1$. Let $W_i = V(\Gamma_i) \cup \{a_i, b\}$. Since $Q[W_i] = \overline{K}_2 + \Gamma_i$ and $\Gamma_i \rightarrow (3, 3)$, by Proposition 4.2 we have $a_i \in V_1$, $i = 1, 2, 3, 4$. Thus V_1 contains the 4-clique $\{a_1, a_2, a_3, a_4\}$, which is a contradiction. □

Since $|V(Q)| = 61$, Proposition 4.3 implies that $F(4, 4; 5) \leq 61$.

5. IDENTIFICATION OF NON-ADJACENT VERTICES

Definition 5.1. Let x, y be two non-adjacent vertices in graph G. Then $G/_{x * y}$ denote the graph G', obtained from G by identifying x and y into new vertex $x * y$, that is, $V(G') = V(G - x - y) \cup \{x * y\}$, $G' - x * y = G - x - y$ and $N_{G'}(x * y) = N_G(x) \cup N_G(y)$. Let $M' = \{x_1 * y_1, \ldots, x_{k-1} * y_{k-1}\}$ and $M = M' \cup \{x_k * y_k\}$, where $x_i, y_i \in V(G)$ and $[x_i, y_i] \notin E(G)$. Then $G/_M = G'/_{x_k * y_k}$, where $G' = G/_{M'}$.

Proposition 5.1. *Let $G \rightarrow (p, q)$. Then $G/_M \rightarrow (p, q)$.*

Proof. It is sufficient to prove that $G_1 = G/_{x_1 * y_1} \rightarrow (p, q)$. Assume that $G_1 \nrightarrow (p, q)$ and let $V(G_1) = V_1 \cup V_2$ be a (p, q)-free 2-coloring. Without a lost of generality we can assume that $x_1 * y_1 \in V_1$. Let $V_1' = (V_1 \setminus \{x_1 * y_1\}) \cup \{x_1, y_1\}$. Then $V_1' \cup V_2$ is a (p, q)-free 2-coloring of $V(G)$, contradicting $G \rightarrow (p, q)$. □

Let G_1 and G_2 be isomorphic graphs without common vertices and let the map $V(G_1) \xrightarrow{\varphi} V(G_2)$ be an isomorphism. Then for $x_1, \ldots, x_k \in V(G_1)$ we define:

$$N_i = \{x_1 * \varphi(x_1), \ldots, x_i * \varphi(x_i)\}, \quad i = 1, \ldots, k;$$

$$\widetilde{G}_i = G_1 \cup G_2/_{N_i}, \quad i = 1, \ldots, k;$$

$$V_1 = V(G_1) \setminus \{x_1, \ldots, x_k\}, \quad V_2 = V(G_2) \setminus \{\varphi(x_1), \ldots, \varphi(x_k)\};$$

$$G' = \widetilde{G}_k[V_1 \cup N_k], \quad G'' = \widetilde{G}_k[V_2 \cup N_k].$$

Proposition 5.2. $e = [x_i * \varphi(x_i), x_j * \varphi(x_j)] \in E(\widetilde{G}_k) \iff [x_i, x_j] \in E(G_1)$.

Proof. If $i = j$, Proposition 5.2 is obvious. Let $i \neq j$ and $j > i$. Clearly, $[x_i, x_j] \in E(G_1)$ implies $e \in E(\widetilde{G}_k)$. Let $[x_i, x_j] \notin E(G_1)$. Then $[\varphi(x_i), \varphi(x_j)] \notin E(G_2)$. Hence, $[x_j, x_i * \varphi(x_i)] \notin E(\widetilde{G}_i)$ and $[\varphi(x_j), x_i * \varphi(x_i)] \notin E(\widetilde{G}_i)$. Thus, $e \notin E(\widetilde{G}_j)$. From $e \notin E(\widetilde{G}_j)$ it follows $e \notin E(\widetilde{G}_k)$. □

Proposition 5.3. (a) *The graphs G' and G'' are isomorphic to the graph G_1;* (b) $\mathrm{cl}(\widetilde{G}_k) = \mathrm{cl}(G_1)$.

Proof. Define the map $\pi : V(G_1) \to V(G')$ as follows:

$$\pi(v) = v, \text{ if } v \in V_1, \quad \text{and} \quad \pi(x_i) = x_i * \varphi(x_i),\ i = 1, \ldots, k.$$

Obviously, π is a bijection. By definition of G', we have

$$[u, v] \in E(G'),\ u, v \in V_1 \iff [u, v] \in E(G_1), \tag{8}$$

$$[u, x_i * \varphi(x_i)] \in E(G'),\ u \in V_1 \iff [u, x_i] \in E(G_1). \tag{9}$$

By Proposition 5.2,

$$[x_i * \varphi(x_i), x_j * \varphi(x_j)] \iff [x_i, x_j] \in E(G_1). \tag{10}$$

From (8), (9) and (10) it follows that π is an isomorphism between G_1 and G'. Similarly, it follows that G_2 and G'' are isomorphic. Since G_1 and G_2 are also isomorphic, Proposition 5.3 (a) follows. Thus, we have

$$\mathrm{cl}(G') = \mathrm{cl}(G'') = \mathrm{cl}(G_1) = \mathrm{cl}(G_2). \tag{11}$$

The proof of Proposition 5.3(b) starts by observing that

$$A \subseteq V(G') \text{ or } A \subseteq V(G'') \text{ for any clique } A \text{ of } \widetilde{G}_k. \tag{12}$$

Assume the opposite. Then there exist $u, v \in A$ such that $u \in V_1$ and $v \in V_2$. By definition of $\widetilde{G}_k$, $[u, v] \notin E(\widetilde{G}_k)$, which is a contradiction. From (12) it follows that $\mathrm{cl}(\widetilde{G}_k) = \mathrm{cl}(G')$ or $\mathrm{cl}(\widetilde{G}_k) = \mathrm{cl}(G'')$. This, together with (11), implies that $\mathrm{cl}(\widetilde{G}_k) = \mathrm{cl}(G_1)$.

6. LEMMAS

Consider the graph $L = \bigcup_{i=1}^{4} \Gamma_i$, where the graphs Γ_i are given in Fig. 1. Define:

$$M_1' = \{u_i^1 * u_i^2,\ i = 1, \ldots, 7\}, \quad M_1'' = \{v_1^1 * v_1^2,\ v_2^1 * v_2^2\}, \quad M_1 = M_1' \cup M_1'';$$
$$M_2' = \{u_i^3 * u_i^4,\ i = 1, \ldots, 7\}, \quad M_2'' = \{v_1^3 * v_1^4,\ v_2^3 * v_2^4\}, \quad M_2 = M_2' \cup M_2'';$$
$$M_3' = \{v_3^1 * v_3^3,\ v_4^1 * v_4^3\}, \quad M_3'' = \{v_5^1 * v_5^3,\ v_6^1 * v_6^3\}, \quad M_3 = M_3' \cup M_3'';$$
$$M_4' = \{v_3^2 * v_3^4,\ v_4^2 * v_4^4\}, \quad M_4'' = \{v_5^2 * v_5^4,\ v_6^2 * v_6^4\}, \quad M_4 = M_4' \cup M_4'';$$
$$M = \bigcup_{i=1}^{4} M_i.$$

Lemma 6.1. *The sets M_i', M_i'', $i = 1, 2, 3, 4$, are independent in graph $L/_M$.*

Proof. Observe that $\{u_1^1, \ldots, u_7^1\}$ is an independent set in Γ_1 and $\{u_1^2, \ldots, u_7^2\}$ is an independent set in Γ_2. Thus, M_1' is an independent set in $L/_M$. Similarly, it follows that the other sets M_i', M_i'' are independent in $L/_M$. □

Lemma 6.2. $\mathrm{cl}\left(L/_M\right) = 3$.

Proof. Define

$$L' = \Gamma_1 \cup \Gamma_2, \quad L'' = \Gamma_3 \cup \Gamma_4, \quad L_1 = L/_{M_1 \cup M_2}.$$

Obviously,

$$L_1 = L'/_{M_1} \cup L''/_{M_2}; \tag{13}$$

$$L/_M = L_1/_{M_3 \cup M_4}. \tag{14}$$

Define the map $V(\Gamma_1) \xrightarrow{\varphi} V(\Gamma_2)$ as follows:

$$v_i^1 \xrightarrow{\varphi} v_i^2, \quad u_i^1 \xrightarrow{\varphi} u_i^2, \quad i = 1, \ldots, 7.$$

Clearly, φ is an isomorphism between Γ_1 and Γ_2. Since $M_1' = \{u_i^1 * \varphi(u_i^1),\ i = 1, \ldots, 7\}$ and $M_1'' = \{v_1^1 * \varphi(v_1^1),\ v_2^1 * \varphi(v_2^1)\}$, from Proposition 4.1 and Proposition 5.3(b) it follows that

$$\mathrm{cl}\left(L'/_{M_1}\right) = 3. \tag{15}$$

Define the map $V\left(L'/_{M_1}\right) \xrightarrow{\psi} V\left(L''/_{M_2}\right)$ as follows:

$$v_i^1 \xrightarrow{\psi} v_i^3, \quad v_i^2 \xrightarrow{\psi} v_i^4, \quad i = 3, \ldots, 7;$$

$$v_1^1 * v_1^2 \xrightarrow{\psi} v_1^3 * v_1^4, \quad v_2^1 * v_2^2 \xrightarrow{\psi} v_2^3 * v_2^4;$$

$$u_i^1 * u_i^2 \xrightarrow{\psi} u_i^3 * u_i^4, \quad i = 1, \ldots, 7.$$

Obviously, ψ is an isomorphism between $L'/_{M_1}$ and $L''/_{M_2}$. Since

$$M_3 = \{v_i^1 * \psi(v_i^1),\ i = 3, \ldots, 6\} \quad \text{and} \quad M_4 = \{v_i^2 * \psi(v_i^2),\ i = 3, \ldots, 6\},$$

from (13) and Proposition 5.3(b) it follows that

$$\mathrm{cl}\left(L_1/_{M_3 \cup M_4}\right) = \mathrm{cl}\left(L'/_{M_1}\right). \tag{16}$$

By (14) – (16), we have $\mathrm{cl}\left(L/_M\right) = 3$.

7. PROOF OF THE THEOREM

I) *Proof of the inequality* $F(4,4;5) \geqq 16$. Let G be a graph such that $G \rightarrow (4,4)$ and $\text{cl}(G) < 5$, i.e. $\text{cl}(G) = 4$. We need to prove that $|V(G)| \geqq 16$. Observe that $|V(G)| \geqq F(4,4;6)$. Since $F(4,4;6) = 14$, [5], we have $|V(G)| \geqq 14$. From $\text{cl}(G) = 4$ and $R(5,3) = 14$, [2], it follows that $\alpha(G) \geqq 3$. Let $\{v_1, v_2, v_3\}$ be an independent set in G. Then $G' = G - \{v_1, v_2, v_3\} \rightarrow (3,4)$ and $\text{cl}(G') = 4$. By $F(3,4;5) = 13$, [8], we have $|V(G')| \geqq 13$. Hence, $|V(G)| \geqq 16$.

II) *Proof of the inequality* $F(4,4;5) \leqq 35$. Consider the graph $R = {}^{Q}/_{M}$, where the graph Q is defined in Section 4 and the set M is given in Section 6. Let $R_1 = R - \{a_1, a_2, a_3, a_4\}$. Observe that

$$R_1 = K_1 + {}^{L}/_{M}, \qquad \text{where } V(K_1) = \{b\} \text{ and } {}^{L}/_{M} \text{ is defined in Section 6.} \tag{17}$$

By Proposition 4.3 and Proposition 5.1, we have $R \rightarrow (4,4)$. We prove that $\text{cl}(R) = 4$. Assume that $\text{cl}(R) \geqq 5$ and let $A \subseteq V(R)$ be a 5-clique of R. By Lemma 6.2, $\text{cl}\left({}^{L}/_{M}\right) = 3$. Since $N_R(b) = V\left({}^{L}/_{M}\right)$, this implies that $b \notin A$. From (17), $\text{cl}\left({}^{L}/_{M}\right) = 3$ and $b \notin A$ it follows that $|V(R_1) \cap A| \leqq 3$. Hence,

$$|A \cap \{a_1, a_2, a_3, a_4\}| \geqq 2. \tag{18}$$

Observe that

$$N_R(a_1) \cap N_R(a_2) = M_1 \cup \{a_3, a_4\} = (M_1' \cup \{a_3\}) \cup (M_1'' \cup \{a_4\}). \tag{19}$$

By Lemma 6.1, M_1' and M_1'' are independent sets. Since $M_1' \cap N_R(a_3) = \emptyset$ and $M_1'' \cap N_R(a_4) = \emptyset$, the sets $M_1' \cup \{a_3\}$ and $M_1'' \cup \{a_4\}$ are also independent sets. Hence $M_1 \cup \{a_3, a_4\}$ contains no 3-cliques. Thus, (19) implies that $\{a_1, a_2\} \nsubseteq A$. Similarly, it follows that $\{a_i, a_j\} \nsubseteq A$, $\forall i \neq j$. This contradicts (18) and proves $\text{cl}(R) = 4$. So, $R \rightarrow (4,4)$ and $\text{cl}(R) = 4$. Since $|V(G)| = 35$, we have $F(4,4;5) \leqq 35$. □

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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Том 96

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LOWER BOUNDS FOR SOME RAMSEY NUMBERS

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For the Ramsey number $R(p_1, \dots, p_r)$, $r \geq 2$, we prove that

$$R(p_1, \dots, p_r) > \big(R(p_1, \dots, p_s) - 1\big)\big(R(p_{s+1}, \dots, p_r) - 1\big),$$

$s \in \{1, \dots, r-1\}$. This inequality generalizes a result obtained by Robertson (Theorem 1) and improves the lower bounds for some Ramsey numbers.

Keywords: Ramsey numbers

2000 MSC : 05D10

Let $p_i \geq 2$, $i = 1, \dots, r$, be integers. An r-edge coloring $\chi = \{1, \dots, r\}$ of the complete graph of n vertices K_n, which does not contain a monochromatic K_{p_i}, in color i for all $i \in \{1, \dots, r\}$, is called a $(p_1, \dots, p_r)$-free r-coloring. The Ramsey number $R(p_1, \dots, p_r)$ is the smallest integer n such that any r-edge coloring of K_n is not $(p_1, \dots, p_r)$-free.

Robertson has proved in [4] the following theorem:

Theorem 1. *Let $r \geq 3$. For any $p_i \geq 3$, $i = 1, \dots, r$, we have*

$$R(p_1, \dots, p_r) > \big((p_1 - 1)R(p_2, \dots, p_r) - 1\big).$$

In the present note we shall prove the following stronger result:

Theorem 2. *Let $p_i \geq 2$, $i = 1, \dots, r$, be integers and $r \geq 2$. Then for any $s \in \{1, \dots, r-1\}$ we have*

$$R(p_1, \dots, p_r) > \big(R(p_1, \dots, p_s) - 1\big)\big(R(p_{s+1}, \dots, p_r) - 1\big).$$

Since $R(p_1) = p_1$, Theorem 2 is a generalization of Theorem 1.

Proof. Put $t = R(p_1, \dots, p_s) - 1$, $l = R(p_{s+1}, \dots, p_r) - 1$ and $m = tl$. Let $V(K_m)$ be the set of vertices of K_m and let $V(K_m) = \bigcup_{i=1}^{l} V_i$, where $|V_i| = t$. Consider a $(p_1, \dots, p_s)$-free edge coloring $\chi_1 = \{1, \dots, s\}$ of K_t and a $(p_{s+1}, \dots, p_r)$-free edge coloring $\chi_2 = \{s+1, \dots, r\}$ of K_l. Let $V(K_l) = \{z_1, \dots, z_l\}$. Define the r-edge coloring $\chi = \{1, \dots, r\}$ of K_m as follows:

1. $\chi(u, v) = \chi_1(u, v)$, if $u, v \in V_i$ for some $i \in \{1, \dots, l\}$;

2. $\chi(u, v) = \chi_2(z_i, z_j)$, if $u \in V_i$, $v \in V_j$, $i \neq j$.

We need to show that $\chi = \{1, \dots, r\}$ is $(p_1, \dots, p_r)$-free. Let $K_{p_i} \leq K_m$. Two cases must be considered:

Case 1. $i \in \{1, \dots, s\}$. If $V(K_{p_i}) \subseteq V_j$ for some $j \in \{1, \dots, l\}$, then K_{p_i} is not monochromatic of color i by the definition of χ_1. Otherwise, there exist $v', v'' \in V(K_{p_i})$ such that $v' \in V_j$, $v'' \in V_k$, $j \neq k$. Then $\chi(v', v'') \geq s+1$ and hence K_{p_i} is not monochromatic of color i.

Case 2. $i \in \{s+1, \dots, r\}$. If there exist $v', v'' \in V(K_{p_i})$ such that $v', v'' \in V_j$ for some $j \in \{1, \dots, l\}$, then $\chi(v', v'') \leq s$. Hence K_{p_i} is not monochromatic of color i. Otherwise, $|V(K_{p_i}) \cap V_j| \leq 1$, $j \in \{1, \dots, l\}$. We may assume that $|V(K_{p_i}) \cap V_j| = 1$ for all $j \in \{1, \dots, p_i\}$. Let $V(K_{p_i}) \cap V_j = v_j$, $j \in \{1, \dots, p_i\}$. Then $V(K_{p_i}) = \{v_1, \dots, v_{p_i}\}$. By the definition of χ_2, there exist $z_k, z_q \in \{z_1, \dots, z_{p_i}\}$ such that $\chi_2(z_k, z_q) \neq i$. Then $\chi(v_k, v_q) = \chi_2(z_k, z_q) \neq i$. Thus K_{p_i} is not monochromatic of color i. This proves Theorem 2.

Some examples. The lower bounds for some Ramsey numbers given in [2] have been improved by Robertson in [4]. In particular, Robertson has proved that $R(4,4,4,4,4) \geq 1372$, $R(5,5,5,5,5) \geq 7329$, $R(6,6,6,6) \geq 5346$, $R(7,7,7,7) \geq 19261$.

Theorem 2 ($s = 2$) implies the following more precise bounds: $R(4,4,4,4,4) \geq 2160$, $R(5,5,5,5,5) \geq 16129$, $R(6,6,6,6) \geq 10202$, $R(7,7,7,7) \geq 41617$.

Remark 1. This note has been submitted for publication in *Electronic Journal of Combinatorics*. The editor-in-chief informed us that it is impossible for such a paper to be published, since the main result (Theorem 2) is announced in [1]. According to [3], this announce is in Chinese and has no proof. Since [4] contains a detailed proof of the special case $s = 1$, we find it appropriate to present a proof of the general case.

Remark 2. Still better bounds for the Ramsey numbers than the ones given above are announced in [3].

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MAXIMAL DEPTHS OF BOOLEAN FUNCTIONS

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Given any Boolean function, there is an upper bound of its depths with respect to arbitrary complete sets of such functions. We prove the algorithmic computability of the largest of these depths.

Keywords: Boolean function, complete set, Post theorem, maximal depth, algorithmic computability

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1. INTRODUCTION

The depths of the Boolean functions are sometimes used for measuring their complexity (especially in the case when parallel computations are considered). Actually, there are at least two variants of the notion of depth. The difference comes from the presence or absence of the possibility to use 0's and 1's "for free" in the computations. The first of these options is chosen for example in [1]. The notion of depth is defined there in Section 1.3 through a corresponding notion of Boolean circuit, and Section 1.4 shows that an approach through Boolean formulas would yield the same values of the depths. The other variant of the notion, also current in the literature, can be defined in a quite similar way, but without the possibility to use the constants 0 and 1 as predecessors of the gates of the considered circuits.

The gates of each Boolean circuit have as their types Boolean functions belonging to some given set, which usually is chosen to be complete.[1] Therefore the

[1] In fact, the notion of completeness also splits into two ones - the weaker notion corresponds to possible using of 0's and 1's "for free", whereas the stronger one corresponds to the case when there is no such possibility.

depth of a function f depends not only on f, but also on the choice of this set of functions. When we compare the depths of two Boolean functions, the result may also depend on the choice in question. For example, the implication has a smaller depth than the equivalence with respect to the set consisting of negation and conjunction, but the situation is the opposite with respect to the set, whose elements are the constant 1, addition modulo 2 and conjunction (of course, if the constants can be used "for free", the constant 1 may be omitted from the second of these sets). To get a complexity measure depending only on the function f, we shall look for the depth of f in the worst case, i.e. in the case when the depth is maximally large.

2. SOME DEFINITIONS

To avoid reasoning about Boolean circuits or Boolean formulas, we shall define the notion of depth (and also of completeness) in another equivalent way. Suppose Ω is a set of Boolean functions[2]. We define infinite sequences $\Omega^{(0)}, \Omega^{(1)}, \Omega^{(2)}, \ldots$ and $\Omega^{[0]}, \Omega^{[1]}, \Omega^{[2]}, \ldots$ of sets of Boolean functions as follows:

- $\Omega^{(0)}$ is the set of all Boolean functions of the form

  $$g(x_1, \ldots, x_n) = x_k, \quad n = 1, 2, 3, \ldots, \quad k = 1, 2, \ldots, n$$

  (*the projection functions*), whereas $\Omega^{[0]}$ consists of these functions and also all constant Boolean functions;

- $\Omega^{(r+1)}$ is obtained by adding to $\Omega^{(r)}$ all functions of the form

  $$g(x_1, \ldots, x_n) = f(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n))$$

  with f belonging to Ω and $g_1, \ldots, g_m$ belonging to $\Omega^{(r)}$, and $\Omega^{[r+1]}$ is obtained similarly, but with a replacement of $\Omega^{(r)}$ by $\Omega^{[r]}$.

We shall call the set Ω *strongly complete* if each Boolean function belongs to $\Omega^{(r)}$ for some non-negative integer r. The set Ω will be called *weakly complete* if each Boolean function belongs to $\Omega^{[r]}$ for some non-negative integer r. Since $\Omega^{(r)} \subseteq \Omega^{[r]}$ for all Ω and r, any strongly complete set is also weakly complete.

Let h be a Boolean function. If Ω is a strongly complete set of Boolean functions, then the smallest r such that $h \in \Omega^{(r)}$ will be called *the strong depth of h with respect to* Ω, and it will be denoted by ${}^sD_\Omega(h)$. Similarly, if Ω is a weakly complete set of Boolean functions, then the smallest r such that $h \in \Omega^{[r]}$ will be called *the weak depth of h with respect to* Ω, and it will be denoted by ${}^wD_\Omega(h)$. We note that ${}^sD_\Omega(h) \geq {}^wD_\Omega(h)$ for any strongly complete set Ω (due to the inclusion $\Omega^{(r)} \subseteq \Omega^{[r]}$).

[2] We shall consider Boolean functions only of a non-zero number of arguments. In particular, the constants will be regarded as such ones too.

There is an easy reduction of the notions of weak completeness and weak depth to the notions of strong completeness and strong depth, respectively.

Lemma 2.1. *Let Ω be a set of Boolean functions, and let Ω' consist of the one-argument constants* 0 *and* 1 *and of all Boolean functions (including the ones from Ω) that have the form*

$$\lambda x_1 \lambda x_2 \ldots \lambda x_n . f(c_{01}, \ldots, c_{0k_0}, x_1, c_{11}, \ldots, c_{1k_1}, x_2, \ldots, x_n, c_{n1}, \ldots, c_{nk_n})$$

with f in Ω and c_{ij} in $\{0,1\}$. Then $\Omega^{[r]} = \Omega'^{(r)}$ for $r = 1, 2, 3, \ldots$

Proof. By induction on r □.

Corollary 2.1. *Let Ω be a set of Boolean functions, and let Ω' be defined as in Lemma* 2.1. *Then Ω is weakly complete iff Ω' is strongly complete, and in such a case the equality*

$$^{w}D_{\Omega}(h) = {}^{s}D_{\Omega'}(h) \tag{2.1}$$

holds for any non-constant Boolean function h.

Of course, the equality (2.1) does not hold for constant functions, since they have weak depth 0 with respect to any weakly complete set Ω, whereas their strong depths with respect to the corresponding set Ω' will be equal to 1.

By a well-known theorem of Emil Post, the strongly complete sets of Boolean functions can be characterized as follows: a set Ω of Boolean functions is strongly complete iff there are in Ω at least one function not preserving 0, at least one function not preserving 1, at least one function that is not self-dual, at least one function that is not monotonically increasing and at least one non-linear function. Hence, by Corollary 2.1, a set Ω of Boolean functions is weakly complete iff there are in Ω at least one function that is not monotonically increasing and at least one non-linear function.

Remark 2.1. Whenever a finite strongly complete set Ω of Boolean functions and a positive integer n are given, one can consecutively find lists of all n-argument functions in the sets $\Omega^{(r)}$ for $r = 0, 1, 2, \ldots$ This can be done thanks to the fact that only n-argument functions from $\Omega^{(r)}$ are used for the generation of the n-argument functions in $\Omega^{(r+1)}$. To find $^{s}D_{\Omega}(h)$ for a given n-argument Boolean function h, it is sufficient to carry out this process until one reaches for the first time a set $\Omega^{(r)}$ containing h as an element. The weak depth of an n-argument Boolean function with respect to a finite weakly complete set of Boolean functions can be found in a similar way.

Although the number $^{s}D_{\Omega}(h)$ depends both on the function h and on the set Ω, this number remains bounded for any fixed h. In fact, the inequality

$$^{s}D_{\Omega}(h) < 2^{2^n} \tag{2.2}$$

holds for any strongly complete set Ω of Boolean functions and any n-argument Boolean function h. To see this, suppose Ω is a strongly complete set of Boolean functions and n is a positive integer. Since there are only 2^{2^n} n-argument Boolean functions, $\Omega^{(0)}$ contains at least one of them, and $\Omega^{(r)}$ is a subset of $\Omega^{(r+1)}$ for any natural number r, it is clear that $\Omega^{(r+1)} \setminus \Omega^{(r)}$ contains no n-argument Boolean function for some r less than 2^{2^n}. Obviously, all n-argument Boolean functions will belong to $\Omega^{(r)}$ for such an r.[3]

The fact we just indicated allows us to give the following definition: for any Boolean function h, the largest of the numbers ${}^sD_\Omega(h)$, where Ω ranges over all strongly complete sets of Boolean functions, will be called *the maximal strong depth of h* and will be denoted by ${}^sD(h)$.

A quite similar reasoning shows that also ${}^wD_\Omega(h)$ remains bounded for any fixed Boolean function h when Ω ranges over all weakly complete sets of Boolean functions. For any Boolean function h, the largest of the corresponding numbers ${}^wD_\Omega(h)$ will be called *the maximal weak depth of h* and will be denoted by ${}^wD(h)$.

Example 2.1. The maximal strong depth of the negation function is equal to 2. In fact, let $h = \lambda x.\overline{x}$. If Ω is an arbitrary set of Boolean functions, then $\lambda x.x$ is the only one-argument function in $\Omega^{(0)}$. Suppose Ω consists of the constant 1, addition modulo 2 and conjunction. Then Ω is strongly complete, and the only one-argument functions in $\Omega^{(1)} \setminus \Omega^{(0)}$ are the two constants, hence ${}^sD_\Omega(h) \geq 2$. It remains to prove that h has a strong depth not greater than 2 with respect to any strongly complete set of Boolean functions. To prove this, suppose that Ω is an arbitrary strongly complete set of Boolean functions. By the Post Theorem, there are functions f_0 and f_1 in Ω such that $f_0(0,\dots,0) = 1$ and $f_1(1,\dots,1) = 0$. The one-argument functions

$$h_0 = \lambda x.f_0(x,\dots,x), \quad h_1 = \lambda x.f_1(x,\dots,x)$$

belong to $\Omega^{(1)}$. Either some of them coincides with h or these functions are the two constants. In the first case h belongs to $\Omega^{(1)}$, hence ${}^sD_\Omega(h) = 1$. In the second one we may consider some function f in Ω that is not monotonically increasing (such a function exists again by the Post Theorem). Then h can be obtained from f by substitution of constants for all its arguments except for one of them. Therefore h belongs to $\Omega^{(2)}$, hence ${}^sD_\Omega(h) \leq 2$.

Example 2.2. The maximal weak depth of the negation function is 1. Indeed, let h be again this function, and Ω be an arbitrary weakly complete set of Boolean functions. Of course, h does not belong to $\Omega^{[0]}$. Since there is a function in Ω that is not monotonically increasing, and h can be obtained from it by substitution of

[3]The set $\Omega^{(0)}$ contains in fact n different n-argument Boolean functions, therefore the above reasoning actually proves the inequality ${}^sD_\Omega(h) \leq 2^{2^n} - n$, which is stronger than (2.2) for $n > 1$. This small strengthening, however, is quite immaterial, since, as Todor Tsankov noticed, the upper bound 2^{2^n} can be replaced by another one which has a much lower order of magnitude. (His reasoning makes use of the disjunctive normal form representation of the Boolean functions.)

constants for all arguments but one, h belongs to $\Omega^{[1]}$. Thus the negation function has a weak depth 1 with respect to any weakly complete set of Boolean functions.

Remark 2.2. It can be shown that ${}^sD(h) \geq {}^wD(h)$ for any Boolean function h. In fact, if we choose a weakly complete set Ω such that ${}^wD_\Omega(h) = {}^wD(h)$ and define the set Ω' as in Lemma 2.1, then Ω' will be strongly complete and we shall have the inequalities ${}^sD(h) \geq {}^sD_{\Omega'}(h) \geq {}^wD(h)$.

The definitions of maximal strong depth and maximal weak depth of a function do not provide us with algorithms for computing these depths, because there are infinitely many strongly complete and infinitely many weakly complete sets of Boolean functions. The existence of such algorithms will be shown in the rest of the paper.

3. ALGORITHMIC COMPUTABILITY OF THE MAXIMAL WEAK DEPTH

We start with the case of the weak depths, because its treatment is much easier, and we have a result in a more finished state for this case.

We shall use the following six weakly complete sets of Boolean functions:

$$\begin{aligned}
\Omega_1 &= \{\, \lambda x.\overline{x},\ \lambda xy.xy \,\}, \\
\Omega_2 &= \{\, \lambda x.\overline{x},\ \lambda xy.x \vee y \,\}, \\
\Omega_3 &= \{\, \lambda xy.x \to y \,\}, \\
\Omega_4 &= \{\, \lambda xy.\overline{xy} \,\}, \\
\Omega_5 &= \{\, \lambda xy.\overline{x \vee y} \,\}, \\
\Omega_6 &= \{\, \lambda xy.\overline{x \to y} \,\}.
\end{aligned}$$

Lemma 3.1. *For any weakly complete set Ω of Boolean functions some of the sets $\Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6$ is a subset of the set $\Omega^{[1]}$.*

Proof. Let Ω be a weakly complete set of Boolean functions. Some non-linear function f surely belongs to Ω, and a two-argument non-linear function g can be obtained from f by substitution of constants for all its arguments except for two of them. The function g will belong to the set $\Omega^{[1]}$ and will have the form

$$g(x, y) = xy \oplus ax \oplus by \oplus c,$$

where a, b, c belong to $\{0, 1\}$, and "$\oplus$" denotes addition modulo 2. Without a loss of generality we may assume that $a \geq b$. If $a = c = 0$, then $g(x, y) = xy$, and from here, taking into account also Example 2.2, we see that $\Omega_1 \subseteq \Omega^{[1]}$. If $a = 0,\ c = 1$, then $g(x, y) = \overline{xy}$, hence $\Omega_4 \subseteq \Omega^{[1]}$. If $a = 1,\ b = c = 0$, then $g(x, y) = \overline{x \to y}$ and therefore $\Omega_6 \subseteq \Omega^{[1]}$. If $a = 1,\ b = 0,\ c = 1$, then $g(x, y) = x \to y$, hence $\Omega_3 \subseteq \Omega^{[1]}$.

If $a = b = 1$, $c = 0$, then $g(x, y) = x \vee y$, hence $\Omega_2 \subseteq \Omega^{[1]}$. Finally, if $a = b = c = 1$, then $g(x, y) = \overline{x \vee y}$, therefore $\Omega_5 \subseteq \Omega^{[1]}$ □.

Lemma 3.2. *Let Ω be a set of Boolean functions, and let Ω' be any subset of the set $\Omega^{[1]}$. Then $\Omega'^{[r]} \subseteq \Omega^{[r]}$ for $r = 0, 1, 2, \ldots$*

Proof. Induction on r □.

Corollary 3.1. *Let Ω and Ω' be weakly complete sets of Boolean functions, and let $\Omega' \subseteq \Omega^{[1]}$. Then ${}^{w}D_{\Omega'}(h) \geq {}^{w}D_{\Omega}(h)$ for any Boolean function h.*

Theorem 3.1. *For any Boolean function h we have the equality*

$$ {}^{w}D(h) = \max\{{}^{w}D_{\Omega_1}(h), {}^{w}D_{\Omega_2}(h), {}^{w}D_{\Omega_3}(h), {}^{w}D_{\Omega_4}(h), {}^{w}D_{\Omega_5}(h), {}^{w}D_{\Omega_6}(h)\}. \quad (3.1) $$

Proof. Let h be an arbitrary Boolean function, and let d be the right-hand side of (3.1). If Ω is any weakly complete set of Boolean functions, then, by Lemma 3.1 and Corollary 3.1, the inequality ${}^{w}D_{\Omega_i}(h) \geq {}^{w}D_{\Omega}(h)$ holds for some $i \in \{1, 2, 3, 4, 5, 6\}$, hence $d \geq {}^{w}D_{\Omega}(h)$. On the other hand, by the choice of d, there is a weakly complete set Ω (some of the sets $\Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6$) such that $d = {}^{w}D_{\Omega}(h)$ □.

Since, by Remark 2.1, the right-hand side of the equality (3.1) is algorithmically computable, the above theorem shows the algorithmic computability of the maximal weak depth.

4. ALGORITHMIC COMPUTABILITY OF THE MAXIMAL STRONG DEPTH

The algorithmic computability of the maximal strong depth will be shown by means of an equality similar to (3.1), namely a finite class $\mathbb{O}$ of finite strongly complete sets of Boolean functions will be indicated such that

$$ {}^{s}D(h) = \max\{{}^{s}D_{\Omega}(h) | \Omega \in \mathbb{O}\} \quad (4.1) $$

for any Boolean function h. We shall call any such class $\mathbb{O}$ *representative for* ${}^{s}D$.

Before we actually indicate a class that is representative for ${}^{s}D$, we shall give the easily provable analogs of Lemma 3.2 and Corollary 3.1 that will be used now.

Lemma 4.1. *Let Ω be a set of Boolean functions, and let Ω' be any subset of the set $\Omega^{(1)}$. Then $\Omega'^{(r)} \subseteq \Omega^{(r)}$ for $r = 0, 1, 2, \ldots$*

Proof. Induction on r □.

Corollary 4.1. *Let Ω and Ω' be strongly complete sets of Boolean functions,* *and let $\Omega' \subseteq \Omega^{(1)}$. Then ${}^sD_{\Omega'}(h) \geq {}^sD_{\Omega}(h)$ for any Boolean function h.*

To show that a finite class of finite strongly complete sets is representative for ${}^sD$, it would be sufficient to ascertain that this class has a property analogous to the property stated in Lemma 3.1.

Lemma 4.2. *Let $\mathbb{O}$ be a finite class of finite strongly complete sets of Boolean functions, and let for any strongly complete set Ω of Boolean functions there be some subset of $\Omega^{(1)}$ belonging to $\mathbb{O}$. Then $\mathbb{O}$ is representative for ${}^sD$.*

Proof. We reason as in the proof of Theorem 3.1. Let h be an arbitrary Boolean function, and let d be the right-hand side of (4.1). If Ω is any strongly complete set of Boolean functions, then, by the assumption of the lemma and by Corollary 4.1, the inequality ${}^sD_{\Omega'}(h) \geq {}^sD_{\Omega}(h)$ holds for some $\Omega' \in \mathbb{O}$, hence $d \geq {}^sD_{\Omega}(h)$. On the other hand, by the choice of d, there is a strongly complete set Ω (some of the sets belonging to $\mathbb{O}$) such that $d = {}^sD_{\Omega}(h)$ $\square$.

Having in mind the above lemma, we shall aim at indicating a class $\mathbb{O}$ that satisfies the assumption of the lemma.

For any m-argument Boolean function f, any positive integer n and any sequence $k_1, k_2, \ldots, k_m$ of numbers from the set $\{1, 2, \ldots, n\}$, the n-argument Boolean function g defined by

$$g(x_1, x_2, \ldots, x_n) = f(x_{k_1}, x_{k_2}, \ldots, x_{k_m})$$

will be called a *projection instance* of f (an *n-ary projection instance* of f). Clearly, the relation of being a projection instance is reflexive and transitive. Obviously, each Boolean function has exactly one unary projection instance. We note also that, for any set Ω of Boolean functions, the set $\Omega^{(1)}$ consists of all projection functions and all projection instances of functions of Ω.

As usually, T_0, T_1, S, M and L will denote, respectively, the class of all Boolean functions preserving 0, the class of all Boolean functions preserving 1, the class of all self-dual Boolean functions, the class of all monotonically increasing ones and the class of all linear ones. We define finite sets ${T_0}^\dagger$, ${T_1}^\dagger$, $S^\dagger$, $M^\dagger$ and $L^\dagger$ of Boolean functions as follows:

- ${T_i}^\dagger$ is the set of the one-argument Boolean functions not belonging to T_i (for $i = 0, 1$);
- $S^\dagger$ is the set of the symmetric two-argument Boolean functions;
- $M^\dagger$ is the set of the three-argument Boolean functions g satisfying the conditions $g(0, 0, 1) = 1$ and $g(1, 0, 1) = 0$;

- $L^\dagger$ consists of all two-arguments functions not belonging to L and all three-argument functions g of the form

$$g(x, y, z) = xy \oplus yz \oplus xz \oplus ax \oplus by \oplus cz \oplus d$$

with coefficients a, b, c, d in $\{0, 1\}$.

Obviously, C and $C^\dagger$ have an empty intersection for $C = T_0, T_1, S, M, L$.

The existence of a class satisfying the assumption of Lemma 4.2 will follow from the next four lemmas.

Lemma 4.3. *Let C be the class T_0 or the class T_1, and f be a Boolean function not belonging to C. Then the unary projection instance of f belongs to $C^\dagger$.*

Proof. Obvious □.

Lemma 4.4. *Each Boolean function not belonging to the class S has a projection instance belonging to $S^\dagger$.*

Proof. Let f be an m-argument function not belonging to S. Then there are $a_1, a_2, \dots, a_m$ in $\{0, 1\}$ such that

$$f(a_1, a_2, \dots, a_m) = f(\overline{a_1}, \overline{a_2}, \dots, \overline{a_m}).$$

We define the function g by the equality

$$g(x_1, x_2) = f(x_{k_1}, x_{k_2}, \dots, x_{k_m}),$$

where

$$k_i = a_i + 1,\ i = 1, 2, \dots, m.$$

Then g is a projection instance of f and the equalities

$$g(0, 1) = f(a_1, a_2, \dots, a_m), \quad g(1, 0) = f(\overline{a_1}, \overline{a_2}, \dots, \overline{a_m})$$

hold, hence $g(0, 1) = g(1, 0)$ □.

Lemma 4.5. *Each Boolean function not belonging to the class M has a projection instance belonging to $M^\dagger$.*

Proof. Let f be an m-argument function not belonging to M. Then there are some j in $\{1, 2, \dots, m\}$ and some $a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_m$ in $\{0, 1\}$ such that

$$f(a_1, \dots, a_{j-1}, 0, a_{j+1}, \dots, a_m) = 1, \ \ f(a_1, \dots, a_{j-1}, 1, a_{j+1}, \dots, a_m) = 0.$$

We define the function g by the equality

$$g(x_1, x_2, x_3) = f(x_{k_1}, x_{k_2}, \dots, x_{k_m}),$$

where

$$k_i = \begin{cases} 1, & \text{if } i = j, \\ a_i + 2, & \text{if } i \in \{1, \ldots, j-1, j+1, \ldots, m\}. \end{cases}$$

Then g is a projection instance of f and the equalities

$$g(x_1, 0, 1) = f(a_1, \ldots, a_{j-1}, x_1, a_{j+1}, \ldots, a_m), \quad x_1 = 0, 1,$$

hold □.

Lemma 4.6. *Each Boolean function not belonging to the class L has a projection instance belonging to $L^\dagger$.*

Proof. To prove the statement of the lemma, it is sufficient to show, for any integer m greater than 2, that each non-linear m-argument Boolean function not belonging to $L^\dagger$ has a non-linear $(m-1)$-argument projection instance.

Let f be a non-linear m-argument Boolean function not belonging to $L^\dagger$. In the case when $m = 3$, we may reason as follows. The representation of the function f as a Zhegalkin polynomial has the form

$$f(x, y, z) = axyz \oplus b_1yz \oplus b_2xz \oplus b_3xy \oplus c_1x \oplus c_2y \oplus c_3z \oplus d,$$

where $a, b_1, b_2, b_3, c_1, c_2, c_3, d$ are fixed elements of the set $\{0, 1\}$, at least one of the numbers a, b_1, b_2, b_3 is not zero, and if $b_1 = b_2 = b_3 = 1$, then also $a = 1$. For all x, y, z in $\{0, 1\}$ we have

$$\begin{aligned} f(x, y, y) &= (a \oplus b_2 \oplus b_3)xy \oplus c_1x \oplus (b_1 \oplus c_2 \oplus c_3)y \oplus d, \\ f(x, y, x) &= (a \oplus b_1 \oplus b_3)xy \oplus (b_2 \oplus c_1 \oplus c_3)x \oplus c_2y \oplus d, \\ f(x, x, z) &= (a \oplus b_1 \oplus b_2)xz \oplus (b_3 \oplus c_1 \oplus c_2)x \oplus c_3z \oplus d. \end{aligned}$$

If we suppose that all two-argument projection instances of f are linear, then we shall have the equalities

$$a \oplus b_2 \oplus b_3 = a \oplus b_1 \oplus b_3 = a \oplus b_1 \oplus b_2 = 0,$$

but they imply the equalities $a = 0$, $b_1 = b_2 = b_3$, and this leads to a contradiction, since some of the numbers a, b_1, b_2, b_3 is not zero. Now suppose that $m > 3$. We again represent $f(x_1, x_2, \ldots, x_m)$ as a non-linear Zhegalkin polynomial. We shall show that its non-linearity will be preserved if we do an appropriate replacement of one of the variables $x_1, x_2, \ldots, x_m$ by another of them. Clearly, the new non-linear polynomial obtained in this way can be used for the definition of a non-linear $(m-1)$-argument projection instance of f.

Case 1. *The Zhegalkin polynomial representing $f(x_1, x_2, \ldots, x_m)$ contains such a non-linear term T that some two distinct variables x_i and x_j are missing in T.* Then the replacement of x_i by x_j will leave the term T unchanged, and all other terms of the polynomial will go into terms distinct from T – they will remain unchanged or will go into terms containing x_j (depending on the absence or the presence of x_i in them). Therefore the polynomial in question will go again into a non-linear Zhegalkin polynomial.

Case 2. *For any non-linear term in the Zhegalkin polynomial representing* $f(x_1, x_2, \ldots, x_m)$, *at most one of the variables* $x_1, x_2, \ldots, x_m$ *is missing in this term.* The case splits into two subcases.

Subcase 2.1. *There is a term* T *in the polynomial such that exactly one of the variables* $x_1, x_2, \ldots, x_m$ *is missing in* T. Let x_i and x_j be two distinct variables occurring in T. Then the replacement of x_i by x_j transforms T into a non-linear term T' with two missing variables, namely x_i and the variable missing in T. It is easily seen that all other terms of the polynomial (if any) will be transformed into terms distinct from T'. In fact, T' could arise only from some term with exactly one missing variable, and that term should not contain the variable missing in T. Hence the polynomial goes again into a non-linear one.

Subcase 2.2. *The term* $x_1 x_2 \ldots x_m$ *is present in the polynomial, and no other non-linear term is present in it.* In this subcase any replacement of some of the variables $x_1, x_2, \ldots, x_m$ by another of them will transform the polynomial again into a non-linear one □.

Let us define now a class $\mathbb{O}$ as follows: $\mathbb{O}$ has as its elements all sets $\{\lambda x.\overline{x}, g, h\}$, where $g \in S^{\dagger}$, $h \in L^{\dagger}$, and all sets $\{\lambda x.0, \lambda x.1, g, h\}$, where $g \in M^{\dagger}$, $h \in L^{\dagger}$. Clearly, $\mathbb{O}$ is a finite class of finite sets of Boolean functions, and, by the Post theorem, all these sets are strongly complete.

Lemma 4.7. *For any strongly complete set* Ω *of Boolean functions there is some subset of* $\Omega^{(1)}$ *belonging to* $\mathbb{O}$.

Proof. Let Ω be an arbitrary strongly complete set of Boolean functions. By the Post theorem and the preceding four lemmas, there are functions g_0, g_1, g_2, g_3, g_4 such that each of them is a projection instance of some function from Ω, hence belongs to $\Omega^{(1)}$, and the conditions $g_0 \in {T_0}^{\dagger}, g_1 \in {T_1}^{\dagger}, g_2 \in S^{\dagger}, g_3 \in M^{\dagger}, g_4 \in L^{\dagger}$ are satisfied. If some of the functions g_0 and g_1 is the negation function, then $\{\lambda x.\overline{x}, g_2, g_4\}$ is a subset of $\Omega^{(1)}$ belonging to $\mathbb{O}$. Otherwise, $\{\lambda x.0, \lambda x.1, g_3, g_4\}$ is such a subset. □.

Theorem 4.1. *The class* $\mathbb{O}$ *is representative for* ${}^{s}D$.

Proof. By the above lemma and Lemma 4.2 □.

Of course, the algorithmic computability of the strong depth is shown by the above theorem in a fully unpractical way, since the class $\mathbb{O}$ we defined is very large. The result can be considerably improved, but this will be probably done in a further publication.

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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(2,3)-GENERATION OF THE GROUPS $\mathrm{PSL}_4(2^m)$

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We prove that the group $\mathrm{PSL}_4(2^m)$, $m > 1$, is (2,3)-generated.

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1. INTRODUCTION

A group G is said to be (2,3)-generated if $G = \langle x, y\rangle$ for some elements x and y of orders 2 and 3, respectively. So far, (2,3)-generation has been proved for a number of series of finite simple groups, for example A_n, $n \neq 6$, 7, 8 (see [2]), $\mathrm{PSL}_2(q)$, $q \neq 9$ [3], $\mathrm{PSL}_3(q)$, $q \neq 4$ (see [1]), and $\mathrm{PSL}_4(q)$, q odd [5]. In a note added in proof to [5], the authors mention that they have recently proved (2,3)-generation for $\mathrm{PSL}_4(q)$ also in the case of even $q > 2$. As we have not been able to find a proof in the literature and as our approach seems to be quite different from that of the authors of [5], here we give a short proof of this fact. Thus we prove the following

Theorem. *The group* $\mathrm{PSL}_4(2^m)$ *is (2,3)-generated for any* $m > 1$.

2. PROOF OF THE THEOREM

Let $G = \mathrm{SL}_4(q) = \mathrm{PSL}_4(q)$, where $q = 2^m$. It is well-known that the group $\mathrm{PSL}_4(2) \cong \mathrm{A}_8$ is not (2,3)-generated, so we assume $m > 1$ in what follows.

The group G acts naturally on a four-dimensional vector space V over the field $\mathrm{GF}(q)$ with a fixed basis e_1, e_2, e_3, e_4. Let ω be a generator of the group $\mathrm{GF}(q^3)^*$ and $\alpha = \omega + \omega^q + \omega^{q^2}$, $\beta = \omega^{1+q} + \omega^{q+q^2} + \omega^{q^2+1}$, $\gamma = \omega^{1+q+q^2}$. Then α, β, $\gamma \in \mathrm{GF}(q)$ and γ has order $q-1$ in the group $\mathrm{GF}(q)^*$, in particular $\gamma \neq 1$ as $q > 2$. The polynomial

$$f(t) = (t+\omega)(t+\omega^q)(t+\omega^{q^2}) = t^3 + \alpha t^2 + \beta t + \gamma$$

is irreducible over $\mathrm{GF}(q)$.

Now, the matrices

$$x = \begin{pmatrix} 0 & \alpha\gamma^{-1} & 1 & \beta \\ 0 & 0 & 0 & \gamma \\ 1 & \beta\gamma^{-1} & 0 & \alpha \\ 0 & \gamma^{-1} & 0 & 0 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

are elements of G of orders 2 and 3, respectively. Let

$$z = xy = \begin{pmatrix} 0 & 1 & \beta & \alpha\gamma^{-1} \\ 0 & 0 & \gamma & 0 \\ 1 & 0 & \alpha & \beta\gamma^{-1} \\ 0 & 0 & 0 & \gamma^{-1} \end{pmatrix}.$$

The characteristic polynomial of z is $(t+\gamma^{-1})f(t)$ and the characteristic roots γ^{-1}, ω, ω^q, ω^{q^2} of z are pairwise distinct. Then, in $\mathrm{GL}_4(q^3)$, z is conjugate to $\mathrm{diag}\,(\gamma^{-1}, \omega, \omega^q, \omega^{q^2})$ and hence z is an element of G of order q^3-1.

Denote $H = \langle x, y\rangle$, $H \leq G$.

Lemma 2.1. *The group H acts irreducibly on the space V.*

Proof. Assume that W is a non-trivial H-invariant subspace of V. Let first $\dim W = 1$ and $W = \langle w\rangle$, $w \neq 0$. Then $x(w) = w$, which yields $w = \mu e_1 + \nu e_2 + \big(\mu + \gamma^{-1}(\alpha+\beta)\nu\big)e_3 + \gamma^{-1}\nu e_4$, μ, $\nu \in \mathrm{GF}(q)$, $\mu \neq 0$ or $\nu \neq 0$. Now $y(w) = \lambda w$, $\lambda \in \mathrm{GF}(q)$, $\lambda^3 = 1$, which produces consecutively $\nu \neq 0$, $\lambda = \gamma^{-1} \neq 1$, whence $\gamma^2+\gamma+1 = 0$, $\mu = 0$, and $\alpha+\beta = \gamma^2$. This yields $f(1) = 1+\alpha+\beta+\gamma = \gamma^2+\gamma+1 = 0$, an impossibility as $f(t)$ is irreducible over $\mathrm{GF}(q)$.

Let $\dim W = 2$. Then the characteristic polynomial of $z_{|W}$ has degree two and must divide the polynomial $(t+\gamma^{-1})f(t)$, again contradicting the irreducibility of $f(t)$.

Lastly, let $\dim W = 3$. The subspace $U = \langle e_1, e_2, e_3\rangle$ of V is $\langle z\rangle$-invariant. Suppose that $W \neq U$. Then $U \cap W$ is a 2-dimensional $\langle z\rangle$-invariant subspace of V, which (as shown above) is impossible. Thus $W = U$, but obviously U is not $\langle x\rangle$-invariant, a contradiction. The lemma is proved. □

Lemma 2.2. *Let M be a maximal subgroup of G having an element of order*

$q^3 - 1$. *Then* M *is the stabilizer of a subspace* W *of* V *with* $\dim W = 1$ *or* 3.

Proof. Suppose false. Then the list of maximal subgroups of G [4] implies that one of the following holds:

1) $|M| = q^6(q-1)^3(q+1)^2$.
2) $|M| = 24(q-1)^3$ if $q > 4$.
3) $|M| = 2q^2(q-1)^3(q+1)^2$.
4) $|M| = 2q^2(q-1)(q+1)^2(q^2+1)$.
5) $M \cong SL_4(q_0)$ if $q = q_0^r$ and r is a prime, $|M| = q_0^6(q_0-1)^3(q_0+1)^2(q_0^2+1)(q_0^2+q_0+1)$.
6) $M \cong Sp_4(q)$, $|M| = q^4(q-1)^2(q+1)^2(q^2+1)$.
7) $M \cong SU_4(q_0)$ if $q = q_0^2$, $|M| = q_0^6(q_0-1)^2(q_0+1)^3(q_0^2+1)(q_0^2-q_0+1)$.

As q^3-1 divides $|M|$ and as $\big(q^2+q+1, 2(q+1)(q^2+1)\big) = 1$, in cases 1), 2), 3), 4), 6) it follows that q^2+q+1 divides $(q-1)^2$, $3(q-1)^2$, $(q-1)^2$, 1, $q-1$, respectively. This is easily seen to be impossible. Similarly, in case 7) it follows that $q_0^2+q_0+1$ divides q_0-1. In case 5), if $r > 2$, then $\big(q^3-1, 2(q_0+1)(q_0^2+1)\big) = 1$ and hence q^3-1 divides $(q_0-1)^3(q_0^2+q_0+1)$. This is impossible as $(q_0-1)^3(q_0^2+q_0+1) < q_0^6 - 1 < q_0^{3r} - 1 = q^3 - 1$. Lastly, in case 5) and $r = 2$, as $\big(q_0^2 - q_0 + 1, 2(q_0-1)(q_0^2+1)\big) = 1$, it follows that $q_0^2 - q_0 + 1$ divides $q_0 + 1$, which yields $q_0 = 2$ and $q = 4$. However, then $M \cong SL_4(2) \cong A_8$ has no element of order $4^3 - 1 = 63$, a contradiction. The lemma is proved. □

We can now complete the proof of the theorem. Assume that $H \neq G$. Let M be a maximal subgroup of G containing H. As M has an element z of order q^3-1, Lemma 2.2 implies that M is the stabilizer of a subspace W of V with $\dim W = 1$ or 3. But then W is H-invariant, which contradicts Lemma 2.1. Thus $H = G$ and $G = \langle x, y\rangle$ is a (2,3)-generated group.

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INSTABILITY OF SOLITARY WAVE SOLUTIONS OF A CLASS OF NONLINEAR DISPERSIVE SYSTEMS[1]

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In this paper the orbital stability and instability properties of solitary wave solutions of a class of nonlinear dispersive systems are studied. By applying the abstract results of Grillakis et al. ([11]), we obtain the stability of the solitary waves.

Keywords: dispersive system, solitary waves, stability

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1. INTRODUCTION

In the present paper we consider the stability and instability of solitary wave solutions $(\varphi(x-ct), \psi(x-ct))$ for the following system of nonlinear evolution equations:

$$\begin{cases} Mu_t + u_x + (u^p v^{p+1})_x = 0 \\ Mv_t + v_x + (u^{p+1} v^p)_x = 0, \end{cases} \tag{1.1}$$

where $u(x,t)$ and $v(x,t)$ are real-valued functions and M is a pseudodifferential operator of order $\mu > 1$ (see (2.1)) and $p > 0$. This system can also be interpreted as a coupled version of the generalized Benjamin-Bona-Mahony (BBM) equation

$$Mu_t + (a(u))_x = 0.$$

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Among the papers devoted to the stability of the BBM equation are [13], [14] and [16]. When $a(u) = u^p$ and $M = 1 - \partial_x^2$, it is obtained in [14] that solitary waves are stable for all p. In [16] this result is extended for a more general class of pseudodifferential operators.

Here, using the same lines of ideas as in [12] and [16], we show that if $p \le \mu$, then solitary waves are always stable, while if $p > \mu$, there is a critical speed c_0 such that we have instability for $c < c_0$ and stability for $c > c_0$.

System (1.1) has four natural invariants $E(u,v) = -\dfrac{2}{p+1}\displaystyle\int u^{p+1}v^{p+1}dx$, $V(u,v) = \dfrac{1}{2}\displaystyle\int [u^2 + v^2 + uMu + vMv]dx$, $I_1(u,v) = \displaystyle\int udx$, $I_2(u,v) = \displaystyle\int vdx$. Our analysis is based on the invariants E and V, following the proofs of [16], [11] and [9].

This paper is organized as follows. In Section 2 we discuss the existence and the asymptotic behavior of solutions of (1.1). In Section 3 we state our main assumptions and prove the stability and instability results.

Notations:

∘ The norm in $H^s(\mathbb{R})$ will be denoted by $||\cdot||_s$, and $||\cdot||$ will denote the norm in $L^2(\mathbb{R})$.

∘ We denote $X^s = H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $X = L^2(\mathbb{R}) \times L^2(\mathbb{R})$ and $||\mathbf{f}||_{X^s} = ||f||_s^2 + ||g||_s^2$ for $\mathbf{f} = (f, g)$.

∘ $\widehat{\Lambda^\mu g}(\xi) = |\xi|^\mu \widehat{g}(\xi)$, $L = \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}$.

2. THE EVOLUTION EQUATION

We begin with a discussion of the existence and uniqueness theory of the initial value problem associated with (1.1). The operator M has the form

$$\widehat{Mu}(\xi) = (1 + |\xi|^\mu)\widehat{u}(\xi). \tag{2.1}$$

We state the basic theorem which guarantees the existence and uniqueness of solutions of (1.1) in $H^{\frac{\mu}{2}}(\mathbb{R})$.

Theorem 2.1. *If* $\mathbf{u}_0 \in X^\nu$, *then there exists a unique global solution* $\mathbf{u}$ *of* (1.1) *in* $C([0,\infty); X^\nu)$ *with* $\mathbf{u}(0) = \mathbf{u}_0$. *Moreover, the functionals* E, V, I_1 *and* I_2 *are constant with respect to* t.

Proof. In order to obtain the existence of weak solutions, we consider the problem

$$\mathbf{u}_t + A\mathbf{u} + G(\mathbf{u}) = 0, \quad \mathbf{u}(0) = \mathbf{u}_0, \tag{2.2}$$

where

$$A = \begin{pmatrix} M^{-1}\partial_x & 0 \\ 0 & M^{-1}\partial_x \end{pmatrix} \text{ and } G(\mathbf{u}) = \begin{pmatrix} M^{-1}\partial_x(u^p v^{p+1}) & 0 \\ 0 & M^{-1}\partial_x(u^{p+1}v^p) \end{pmatrix}.$$

Equation (2.2) can be written as an integral equation

$$\mathbf{u} = U(t)\mathbf{u}_0 + \int_0^t U(t-\tau)G(\mathbf{u}(\tau))d\tau,$$

where $U(t)$ is a C_0 group of unitary operators in X^ν generated by a skew adjoint operator A with $D(A) = X^\nu$ and $\mathbf{u}_0 \in D(A)$. We solve the integral equation by the semigroup theory. Since $X^\nu \subset L^\infty \times L^\infty$, it is easy to show that $\mathbf{u} \to G(\mathbf{u})$ carries $Y \to Y$ in locally Lipschitzian manner, where Y denotes a Hilbert product space of $D(A)$ with the graph norm given by $||\mathbf{u}||_Y = ||\mathbf{u}||_{X^\nu} + ||A\mathbf{u}||_{X^\nu}$. By [15], Theorem 6.1.4, for any $\mathbf{u}_0 \in X^\nu$ there is some $T \in (0,\infty)$ so that a unique solution $\mathbf{u}(\cdot,t)$ with initial data $\mathbf{u}_0$ exists for $0 < t \leq T$.

Multiplying (1.1) by (u,v) yields

$$\frac{d}{dt}||\mathbf{u}(t)||_{X^\nu} = 0.$$

This implies that $\mathbf{u}$ is bounded in X^ν and proves the global existence of a weak solution $\mathbf{u}$ for (1.1).

The fact that E and V are constants follows from the local existence. Finally, if $I_1(u_0,v_0)$ and $I_2(u_0,v_0)$ exist, then $I_1(u(t),v(t))$ and $I_2(u(t),v(t))$ do exist and $I_1(u_0,v_0) = I_1(u(t),v(t))$ and $I_2(u_0,v_0) = I_2(u(t),v(t))$. This follows by integrating each equation of (1.1) over (a,b) and letting $a \to -\infty$, $b \to \infty$. This completes the proof of Theorem 2.1. □

Consider the linear initial value problem associated to Eq. (1.1)

$$\begin{cases} Mu_t + u_x = 0 \\ Mv_t + v_x = 0 \\ (u(0),v(0)) = (u_0,v_0) \in X^\nu \end{cases} \tag{2.3}$$

and the related unitary group $V(t)$ which is defined by

$$V(t)f(x) = S_t \star f(x),$$

where S_t is defined by the oscillatory integral

$$S_t(x) = \int_{-\infty}^{\infty} e^{it(\frac{\xi}{1+|\xi|^\mu} - x\xi)} d\xi.$$

Therefore the solution of Eq. (2.3) is given by the unitary group $W(t)$ in X^ν defined for $\mathbf{u}_0 = (u_0,v_0)$ by

$$W(t)\mathbf{u}_0 = (V(t)u_0(x), V(t)v_0(x)).$$

Theorem 2.2. *Let* $\mathbf{u} \in X^\nu \cap (L^1(\mathbb{R}) \times L^1(\mathbb{R}))$ *and let* $\mathbf{u}(x,t)$ *be the solution of* (1.1) *with initial data* $\mathbf{u}_0$. *Then there exists* $0 < \eta < 1$ *such that*

$$\sup_{-\infty<z<\infty} \left| \int_{-\infty}^{z} \mathbf{u}(x,t)dx \right| \le c(1+t^\eta), \tag{2.4}$$

where the constant c *depends only on* $\mathbf{u}_0$.

To prove Theorem 2.2, we need the following lemma, which is proved in [16].

Lemma 2.1. *Let* $S(t)$ *be the evolution operator to the linear equation*

$$((1+\Lambda^\mu)\partial_t + \partial_x)w = 0 \quad (S(t)w(0) = w(t)).$$

Then $S(t) : H^\nu \cap L^1 \to L^\infty$ *for all* $t > 0$. *Moreover, there exist* $\theta \in (0,1)$ *and* $c > 0$ *such that*

$$|S(t)w_0|_\infty \le ct^{-\theta}(|w_0|_1 + ||w_0||_\nu), \quad \theta = \frac{\mu-1}{2\mu}.$$

From Lemma 2.1 and Young's inequality for convolution we have

$$|W(t)|_{L^\infty \times L^\infty} \le ct^{-\theta}(|\mathbf{u}_0|_{L^1\times L^1} + ||\mathbf{u}_0||_{X^\nu}). \tag{2.5}$$

Proof of Theorem 2.2. Let $\mathbf{z}(t) = W(t)\mathbf{u}_0$, that is

$$L\partial_t \mathbf{z} + \partial_x \mathbf{z} = 0, \quad \mathbf{z}(0) = \mathbf{u}_0.$$

Then

$$\begin{aligned} \mathbf{u}(t) &= \mathbf{z}(t) - \int_0^t W(t-\tau)L^{-1}\partial_x F(\mathbf{u})d\tau \\ &= \mathbf{z}(t) - \partial_x \int_0^t W(t-\tau)L^{-1}F(\mathbf{u})d\tau, \end{aligned}$$

where $F(\mathbf{u}) = (u^p v^{p+1}, u^{p+1}v^p)$.
Let $U(x,t) = \int_{-\infty}^x \mathbf{u}(y,t)dy$ and $Z(x,t) = \int_{-\infty}^x \mathbf{z}(y,t)dy$. Then

$$U(t) = Z(t) - \int_0^t W(t-\tau)L^{-1}F(\mathbf{u})d\tau. \tag{2.6}$$

We estimate the two terms on the right-hand side of (2.6) separately. First, we obtain from the equation for $\mathbf{z}(x,t)$,

$$\mathbf{z}(t) = \mathbf{u}_0 - \partial_x \int_0^t L^{-1}\mathbf{z}(\tau)d\tau,$$

so that

$$Z(T) = U_0 - \int_0^t W(\tau)L^{-1}\mathbf{u}_0 d\tau$$

with $U_0(x) = \int_{-\infty}^{x} \mathbf{u}_0(y)dy$. Using (2.5), we have

$$|Z(x,t)| \leq |\mathbf{u}_0|_{L^1\times L^1} + c\int_0^t (1+\tau)^{-\theta}d\tau(|L^{-1}\mathbf{u}_0|_{L^1\times L^1} + ||L^{-1}\mathbf{u}_0||_{X^\nu})$$

$$\leq c(1+t)^{\eta}(|L^{-1}\mathbf{u}_0|_{L^1\times L^1} + ||L^{-1}\mathbf{u}_0||_{X^\nu}),$$

where $\eta = 1-\theta$. Noticing that $||L^{-1}\mathbf{u}_0||_{X^\nu} \leq c||\mathbf{u}_0||_{X^\nu}$, then

$$|Z(x,t)| \leq c(1+t)^{\eta}(|\mathbf{u}_0|_{L^1\times L^1} + ||\mathbf{u}_0||_{X^\nu}).$$

Let $P(x,t)$ denote the second term on the right-hand side of Eq. (2.6). Then by (2.5)

$$|P(x,t)| \leq \left|\int_0^t W(t-\tau)L^{-1}F(\mathbf{u})d\tau\right|$$

$$\leq c\int_0^t (1+t-\tau)^{-\theta}d\tau(|L^{-1}F(\mathbf{u})|_{L^1\times L^1} + ||L^{-1}F(\mathbf{u})||_{X^\nu}).$$

Since $X^\nu \subset L^\infty \times L^\infty \quad (\nu > \frac{1}{2})$, then $|L^{-1}F(\mathbf{u})|_{L^1\times L^1}$ is bounded uniformly in τ by a constant which depends only on $\mathbf{u}_0$. Next observe that $||L^{-1}F(\mathbf{u})||_{X^\nu} \leq (|u|_\infty^p + |v|_\infty^p)||\mathbf{u}||_{X^\nu}$. Thus

$$|P(x,t)| \leq c(1+t)^{\eta}.$$

This completes the proof of the theorem. □

3. THE SOLITARY WAVE

We consider a smooth solution of (1.1) of the form $(u(x,t), v(x,t)) = (\varphi(x-ct), \psi(x-ct)) = \Phi(x-ct)$ that vanishes at infinity. Substituting Φ in (1.1) and assuming that $\varphi, \psi, \varphi', \psi', \varphi'', \psi'' \to 0$ as $|\zeta| \to \infty$, we obtain

$$\begin{cases} -cM\varphi + \varphi + \varphi^p\psi^{p+1} = 0 \\ -cM\psi + \psi + \varphi^{p+1}\psi^p = 0. \end{cases} \tag{3.1}$$

From (3.1) we see that if E' and V' represent the Frechet derivatives of E, V, then

$$E'(\varphi_c, \psi_c) + cV'(\varphi_c, \psi_c) = 0. \tag{3.2}$$

Moreover, if H_c is the linearized operator of $E' + cV'$ around Φ_c, namely

$$H_c = E''(\Phi_c) + cV''(\Phi_c) \tag{3.3}$$

$$= \begin{pmatrix} c\Lambda^\mu + (c-1) - p\varphi^{p-1}\psi^{p+1} & -(p+1)\varphi^p\psi^p \\ -(p+1)\varphi^p\psi^p & c\Lambda^\mu + (c-1) - p\varphi^{p+1}\psi^{p-1} \end{pmatrix};$$

then $H_c(\partial_x\varphi_c, \partial_x\psi_c) = 0$.
We now establish our main assumptions on Φ_c and H_c under which we solve the problem of stability and instability. They are as follows.

Assumption 1. There is an interval $(c_1, c_2) \subset \mathbb{R}$ such that for every $c \in (c_1, c_2)$ there exists a solution $\Phi_c = (\varphi_c, \psi_c)$, $\varphi > 0$, $\psi > 0$ of (3.2) in $X^{\nu+3}$. The curve $c \to \Phi_c$ is C^1 with values in $X^{\nu+1}$. Moreover, $(1+|\xi|)^{\frac{1}{2}}\dfrac{d\Phi_c}{dc} \in L^1 \times L^1$.

Assumption 2. The zero eigenvalue of the operator H_c is simple. H_c has a unique negative simple eigenvalue with an eigenfunction χ_c. Besides the negative and the zero eigenvalues, the rest of the spectrum of H_c is positive and bounded away from zero. Moreover, the mapping $c \to \chi_c$ is continuous with values in $X^{\nu+1}$ and $(1+|\xi|)^{\frac{1}{2}}\chi_c \in L^1 \times L^1$, $\chi_1 > 0$, $\chi_2 > 0$.

Denote
$$d(c) = E(\Phi_c) + cV(\Phi_c).$$
After a differentiation with respect to c, we have
$$d'(c) = \langle E'(\Phi_c) + cV'(\Phi_c), \frac{d\Phi_c}{dc}\rangle + V(\Phi_c) = V(\Phi_c), \tag{3.4}$$
$$d''(c) = \langle V'(\Phi_c), \frac{d\Phi_c}{dc}\rangle. \tag{3.5}$$
Next we examine the relation between the convexity properties of the function $d(c)$ and the properties of the functional E near the critical point Φ_c under the constraint $V = \text{const}$.

Theorem 3.3. *Let $c > 0$ be fixed. If $d''(c) < 0$, then there is a curve $w \to \Psi_w$ which satisfies $V(\Phi_c) = V(\Psi_w)$. $\Phi_c = \Psi_c$, and on which $E(\mathbf{u})$ has a strict local maximum at $\mathbf{u} = \Phi_c$.*

Proof. Following the ideas of Souganidis and Strauss [16], we note that for $G(w,s) = V(\Phi_w + s\chi_c)$, $G(c,0) = V(\Phi_c)$ and $\dfrac{\partial}{\partial s}V(\Phi_w + s\chi_c)(c,0) = \langle V'(\Phi_c), \chi_c\rangle = (L\Phi_c, \chi_c) \neq 0$. Therefore, it follows from the implicit function theorem that there is a C^1 function $s(w)$ for w near c such that $s(c) = 0$ and $G(w, s(w)) = V(\Phi_c)$. Next we define $\Psi_w = \Phi_c + s(w)\chi_x$. It is easy to be seen that $\dfrac{d}{dw}E(\Psi_w)_{|w=c} = 0$ and
$$\frac{d^2}{dw^2}E(\Psi_w)_{|w=c} = \langle H_c\mathbf{y}, \mathbf{y}\rangle,$$
where $\mathbf{y} = \dfrac{d\Psi_w}{dw}_{|w=c} = \dfrac{d}{dc}\Phi_c + s'(c)\chi_c$. So it suffices to show that $\langle H_c\mathbf{y}, \mathbf{y}\rangle < 0$.
We have
$$0 = \frac{d}{dw}V(\Psi_w)_{|w=c} = \langle V'(\Phi_c), \frac{d}{dw}_{|w=c}\rangle$$

$$= (L\Phi_c, \mathbf{y}) = (L\Phi_c, \frac{d}{dc}\Phi_c) + s'(c)(L\Phi_c, \chi_c).$$

From (3.5), $d''(c) = -s'(c)(L\Phi_c, \chi_c)$, therefore

$$(H_c\mathbf{y}, \mathbf{y}) = s'(c)(H_c\chi_c, \mathbf{y}) - (L\Phi_c, \mathbf{y}) = d''(c) + s'^2(c)(H_c\chi_c, \chi_c) < 0.$$

This proves the theorem. □

We continue our study by specifying the precise form in which stability and instability are to be interpreted. Denoting by τ_s, $s \in \mathbb{R}$, the translation operator $\tau_s f(x) = f(x+s)$ for all $x \in \mathbb{R}$, we define $T(s)\mathbf{f} = (\tau_s f, \tau_s g)$ for $\mathbf{f} = (f, g)$. For $\varepsilon > 0$ consider the tubular neighborhood

$$U_\varepsilon = \{\mathbf{g} \in X^\nu \mid \inf_{s\in\mathbb{R}} ||\mathbf{g} - T(s)\Phi_c||_{X^\nu} < \varepsilon\}.$$

Definition 3.1. The solitary wave Φ_c is X^ν stable if for every $\varepsilon > 0$ there is $\delta > 0$ such that if $\mathbf{u}_0 \in U_\delta$, then (1.1) has a unique solution $\mathbf{u}(t) \in C([0,\infty); X^\nu)$ with $\mathbf{u}(0) = \mathbf{u}_0$ and $\mathbf{u}(t) \in U_\varepsilon$ for all $t \in \mathbb{R}$. Otherwise, Φ_c is called unstable.

The stability assertion (when $d''(c) > 0$) is a special case of [11], so that we omit the proof. For the instability, we need a series of preliminary results which can be proved as in the analogous cases of [9]. For this reason we only state them without proof.

Lemma 3.2. *There are an $\varepsilon > 0$ and a unique C^1 map $\alpha : U_\varepsilon \to \mathbb{R}$ such that for $\mathbf{u} \in U_\varepsilon$ and $r \in \mathbb{R}$:*

(i) $\langle \mathbf{u}(\cdot + \alpha(\mathbf{u})), \partial_x\Phi_c\rangle = 0, \qquad \alpha(\Phi_c) = 0;$

(ii) $\alpha(\mathbf{u}(\cdot + r)) = \alpha(\mathbf{u}) - r;$

(iii) $\alpha'(\mathbf{u}) = \dfrac{\partial_x\Phi_c(\cdot - \alpha(\mathbf{u}))}{\langle \mathbf{u}, \partial_x^2\Phi_c(\cdot - \alpha(\mathbf{u}))\rangle}.$

Next we define an auxiliary operator B which will play a crucial role in the proof of instability. If $\mathbf{y}$ is as in Theorem 3.3, then $(H_c\mathbf{y}, \mathbf{y}) < 0$ and $(\mathbf{y}, L\Phi_c) = 0$.

Definition 3.2. For $\mathbf{u} \in U_\varepsilon$, define $B(\mathbf{u})$ by the formula

$$B(\mathbf{u}) = \mathbf{y}(\cdot - \alpha(\mathbf{u})) - \frac{(L\mathbf{u}, \mathbf{y}(\cdot - \alpha(\mathbf{u})))}{\langle \mathbf{u}, \partial_x^2\Phi_c(\cdot - \alpha(\mathbf{u}))\rangle} L^{-1}\partial_x^2\Phi_c(\cdot - \alpha(\mathbf{u})).$$

Lemma 3.3. *B is a C^1 function from U_ε into X^ν. Moreover, B commutes with translations, $B(\Phi_c) = \mathbf{y}$ and $\langle B(\mathbf{u}), L\mathbf{u}\rangle = 0$ for every $\mathbf{u} \in U_\varepsilon$.*

Lemma 3.4. *There is a C^1 functional $\Upsilon : D_\varepsilon \to \mathbb{R}$, where $D_\varepsilon = \{\mathbf{v} \in U_\varepsilon : V(\mathbf{v}) = V(\Phi_c)\}$, such that if $\mathbf{v} \in D_\varepsilon$ and $\mathbf{v}$ is not a translate of Φ_c, then*

$$E(\Phi_c) < E(\mathbf{v}) + \Upsilon(\mathbf{v})\langle E'(\mathbf{v}), B(\mathbf{u})\rangle.$$

Lemma 3.5. *The curve $w \to \Psi_w$, constructed in Theorem 3.3, satisfies $E(\Psi_w) < E(\Phi_c)$ for $w \neq c$, $V(\Psi_w) = V(\Phi_c)$ and $\langle E'(\Psi_w), B(\Psi_w)\rangle$ changes its sign as w passes through c.*

Theorem 3.4. *Assume that Assumptions 1 and 2 hold and $d''(c) < 0$. Then the solitary wave Φ_c is unstable.*

Proof. Let $\varepsilon > 0$ be small enough such that Lemma 3.2 and its consequences apply with U_ε. To prove instability of Φ_c, it suffices to show that there are some elements $\mathbf{u}_0 \in X^\nu$ which are arbitrary close to Φ_c, but for which the solution $\mathbf{u}$ of Eq. (1.1) with initial data $\mathbf{u}_0$ leaves U_ε in finite time.

By Lemma 3.5, we can find $\mathbf{u}_0 \in X^\nu$ which is close to Φ_c and satisfies $V(\mathbf{u}_0) = V(\Phi_c)$, $E(\mathbf{u}_0 < E(\Phi_c)$ and $\langle E'(\mathbf{u}_0), B(\mathbf{u}_0)\rangle > 0$. For a fixed $\mathbf{u}_0$, let $[0, t_1)$ denote the maximal interval for which $\mathbf{u}(\cdot, t)$ lies continuously in U_ε. It suffices to show that $t_1 < \infty$.

In view of Theorems 2.1 and 2.2 $\mathbf{u}$ has the following properties:

$$\mathbf{u} \in C([0,t_1); X^\nu), \quad \mathbf{u}(x,0) = \mathbf{u}_0,$$

$$\sup_{x\in\mathbf{R}} \left| \int_{-\infty}^{x} \mathbf{u}(z,t)dz \right| \leq c_0(1+t^\eta), \quad t \in [0,t_1),$$

$$\sup_{t\in[0,t_1)} ||\mathbf{u}(t)||_{X^\nu} \leq c_1.$$

Let us take $\beta(t) = \alpha(\mathbf{u}(t))$, $\mathbf{Y}(x) = \int_{-\infty}^{x} L\mathbf{y}(\rho)d\rho = \int_{-\infty}^{x} \mathbf{y}(\rho)d\rho + N\mathbf{y}(x)$, where $N = \begin{pmatrix} \frac{|\xi|^\mu}{i\xi} & 0 \\ 0 & \frac{|\xi|^\mu}{i\xi} \end{pmatrix}$, and define

$$A(t) = \int_{-\infty}^{\infty} \mathbf{Y}(x - \beta(t))\mathbf{u}(x,t)dx. \tag{3.6}$$

Let H be the Heaviside function and $\gamma = \int_{-\infty}^{\infty} \mathbf{u}_0(x)dx$. We note that by Assumptions 1 and 2, $\int_{-\infty}^{\infty} (1+|x|)^{\frac{1}{2}}|\mathbf{y}(x)|dx < \infty$ and the function $R(x) = \int_{-\infty}^{\infty} \mathbf{y}(\rho)d\rho - \gamma H(x)$ belongs to $L^2 \times L^2$. Therefore we obtain from Eq. (3.6) that

$$A(t) = \int_{-\infty}^{\infty} R(x-\beta(t))\mathbf{u}(x,t)dx + \gamma\int_{\beta(t)}^{\infty} \mathbf{u}(x,t)dx + \int_{-\infty}^{\infty} N\mathbf{y}(x-\beta(t))\mathbf{u}(x,t)dx.$$

Hence,

$$|A(t)| \leq |R|_2||\mathbf{u}||_{X^\nu} + (c_0(1+t^\eta) + ||N\mathbf{u}||_{X^\nu}|||\mathbf{u}||_{X^\nu}. \tag{3.7}$$

Now

$$\begin{aligned}\frac{dA}{dt} &= -\langle\alpha'(\mathbf{u}), \frac{d\mathbf{u}}{dt}\langle L\mathbf{y}, \mathbf{u}\rangle + \langle \mathbf{Y}(\cdot - \beta), \frac{d\mathbf{u}}{dt}\rangle \\ &= \langle -\langle \mathbf{y}(\cdot - \beta), L\mathbf{u}\rangle\alpha'(\mathbf{u}) + \mathbf{Y}(\cdot - \beta), \frac{d\mathbf{u}}{dt}\rangle\end{aligned}$$

$$= -\langle B(\mathbf{u}), E'(\mathbf{u})\rangle.$$

As $0 < E(\Phi_c) - E(\mathbf{u}_0) = E(\Phi_c) - E(\mathbf{u})$, Lemma 3.3 implies that

$$0 < \Upsilon(\mathbf{u})\langle E'(\mathbf{u}(t)), B(\mathbf{u}(t))\rangle.$$

Moreover, since $\mathbf{u}(t) \in U_\varepsilon$ and $\Upsilon(\Phi_c) = 0$, we may assume that $\Upsilon(\mathbf{u}(t)) < 1$ by choosing ε even smaller if necessary.

Therefore for all $t \in [0, t_1)$, $\langle E'(\mathbf{u}(t)), B(\mathbf{u}(t))\rangle > E(\Phi_c) - E(\mathbf{u}_0) > 0$. Hence for $0 < t < t_1$

$$-\frac{dA}{dt} \geq E(\Phi_c) - E(\mathbf{u}_0) > 0.$$

Comparing this with (3.7), we conclude that $t_1 < \infty$. □

Lemma 3.6. *One has* $d(c) = \frac{\mu c}{2}[\langle \Lambda^\mu \varphi_c, \varphi_c\rangle + \langle \Lambda^\mu \psi_c, \psi_c\rangle]$.

Proof. For $\lambda > 0$, let $\Phi^\lambda(x) = \Phi(\frac{x}{\lambda})$. Then

$$E(\Phi^\lambda) + cV(\Phi^\lambda) = \int [-F(\Phi^\lambda) + \frac{c}{2}\Phi^\lambda L\Phi^\lambda]dx$$

$$= \int [-F(\Phi^\lambda) + \frac{c}{2}(\Phi^\lambda)^2 + \frac{c}{2}\Phi^\lambda \Lambda^\mu \Phi^\lambda]dx$$

$$= \int \lambda[-F(\Phi) + \frac{c}{2}\Phi^2]dx + \lambda^{1-\mu}\frac{c}{2}\int \Phi\Lambda^\mu\Phi dx.$$

Next we differentiate this expression with respect to λ and evaluate it at $\lambda = 1$, observing that the left-hand side becomes zero, because $E'(\Phi^\lambda) + cV'(\Phi^\lambda) = 0$. Thus

$$0 = \int [-F(\Phi) + \frac{c}{2}\Phi^2 + (1-\mu)\frac{c}{2}\Phi\Lambda^\mu\Phi]dx,$$

so that

$$d(c) = \frac{\mu c}{2}\int \Phi\Lambda^\mu\Phi dx.$$

Theorem 3.5. *Let Assumptions* 1 *and* 2 *hold:*

a) *if* $p \leq \mu$, *then* Φ_c *is stable for all* $c > 1$;

b) *if* $p > \mu$, *there is a* $c_0 > 1$ *such that* Φ_c *is stable for* $c > c_0$ *and unstable for* $1 < c < c_0$.

Proof. Using the homogeneity of M, we can write the solution Φ_c as

$$\varphi(x) = (c-1)^{\frac{1}{2p}}\varphi_1\left(\left(\frac{c-1}{c}\right)^{\frac{1}{\mu}} x\right),$$

$$\psi(x) = (c-1)^{\frac{1}{2p}}\psi_1\left(\left(\frac{c-1}{c}\right)^{\frac{1}{\mu}} x\right),$$

where (φ_1, ψ_1) is a solution of the system

$$\Lambda^\mu \varphi_1 + \varphi_1 - \varphi_1^p \psi_1^{p+1} = 0$$

$$\Lambda^\mu \psi_1 + \psi_1 - \varphi_1^{p+1} \psi_1^p = 0,$$

which is independent on c. Then

$$d(c) = \frac{\mu c}{2}\left[\int \varphi \Lambda^\mu \varphi + \int \psi \Lambda^\mu \psi\right]$$

$$= \frac{\mu b}{2}(c-1)^{\frac{1}{p}+1-\frac{1}{\mu}} c^{\frac{1}{\mu}},$$

where $b = \int \varphi_1 \Lambda^\mu \varphi_1 + \int \psi_1 \Lambda^\mu \psi_1$. Differentiating twice with respect to c yields

$$d''(c) = \frac{\mu b}{2}(c-1)^{\frac{1}{p}-\frac{1}{\mu}-1} c^{\frac{1}{\mu}-2} q(c),$$

where $q(c) = (r+s+1)(r+s+2)c^2 - 2(r+1)(r+s+1)c + r(r+1)$, $r = \dfrac{1}{\mu} - 1$, $s = \dfrac{1}{p} - \dfrac{1}{\mu}$. Whether $d''(c)$ is positive or negative depends on the sign of $q(c)$. This is a quadratic function of c with one negative and one positive root, since $r(r+1) < 0$ and $r + s + 1 > 0$. We call the positive root c_0. Since $q(1) = (\dfrac{1}{p} - \dfrac{1}{\mu})(\dfrac{1}{p} - \dfrac{1}{\mu} + 1)$, then if $p \leq \mu$, $d''(c) > 0$ for $c > 1$, and if $p > \mu$, $d''(c) < 0$ for $1 < c < c_0$ and $d''(c) > 0$ for $c > c_0$. Theorem 3.3 is proved. □

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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ESTIMATES FOR THE SINGULAR SOLUTIONS OF THE 3-D PROTTER'S PROBLEM

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For the wave equation we study boundary value problems, stated by Protter in 1952, as some three-dimensional analogues of Darboux problems on the plane. It is known that Protter's problems are not well posed and the solution may have singularity at the vertex O of a characteristic cone, which is a part of the domain's boundary $\partial\Omega$. It is shown that for n in $\mathbb{N}$ there exists a right-hand side smooth function from $C^n(\bar{\Omega})$, for which the corresponding unique generalized solution belongs to $C^n(\bar{\Omega}\backslash O)$, but it has a strong power-type singularity. It is isolated at the vertex O and does not propagate along the cone. The present article gives some necessary and sufficient conditions for the existence of a fixed order singularity. It states some exact a priori estimates for the solution.

Keywords: wave equation, boundary value problems, generalized solution, singular solutions, propagation of singularities, special functions

2000 MSC: main 35L05, 35L20, 35D05, 35A20; secondary 33C05, 33C90

1. INTRODUCTION

We discuss some boundary value problems for the wave equation

$$\Box u = u_{x_1x_1} + u_{x_2x_2} - u_{tt} = f \tag{1.1}$$

in a simply connected domain $\Omega \subset \mathbb{R}^3$. The domain

$$\Omega := \left\{(x_1, x_2, t) : \ 0 < t < 1/2, \ t < \sqrt{x_1^2 + x_2^2} < 1 - t\right\}$$

is bounded by two characteristic cones of (1.1)

$$S_1 = \{(x_1,x_2,t):\ 0<t<1/2,\ \sqrt{x_1^2+x_2^2}=1-t\},$$
$$S_2 = \{(x_1,x_2,t):\ 0<t<1/2,\ \sqrt{x_1^2+x_2^2}=t\}$$

and the circle $S_0=\{(x_1,x_2,t):\ t=0,\ x_1^2+x_2^2<1\}$ centered at the origin $O(0,0,0)$. The following three-dimensional analogues of the plane Darboux problems are stated by M. Protter [27]:

Problems P1 and P2. To find a solution of the wave equation (1.1) in Ω, which satisfies one of the following boundary conditions:

P1 $\quad u|_{S_0}=0$ and $u|_{S_1}=0$;

P2 $\quad u_t|_{S_0}=0$ and $u|_{S_1}=0$.

The corresponding adjoint problems are:

Problems P1* and P2*. To find a solution of the wave equation (1.1) in Ω, which satisfies the corresponding boundary conditions:

P1* $\quad u|_{S_0}=0$ and $u|_{S_2}=0$;

P2* $\quad u_t|_{S_0}=0$ and $u|_{S_2}=0$.

For the recent known results concerning Protter's problems see [25] and references therein. For further publications in this area see [1, 2, 8, 13, 16, 19, 20].

Substituting the boundary condition on S_0 by $[u_t+\alpha u]|_{S_0}=0$, one obtains Problem P_α, for which we refer to [11] and references therein. In the case of the wave equation, involving either lower order terms or some other type perturbations, Problem $P2$ in Ω has been studied in [1, 2, 3, 12]. On the other hand, Bazarbekov [5] gives another analogue of the classical Darboux problem in the same domain Ω. Some other statements of Darboux type problems can be found in [4, 6, 18] in bounded or unbounded domains different from Ω.

Protter [27] formulated and studied these three-dimensional analogues of the Darboux problem on the plane 50 years ago – in 1952. Nowadays, it is known that in contrast to the Darboux problem in $\mathbb{R}^2$ the $3-D$ Problems $P1$ and $P2$ are not well posed. The reason for this is that the adjoint homogeneous Problems $P1^*$ and $P2^*$ have smooth solutions and the linear space they generate is infinite dimensional as one could see in Tong Kwang-Chang [29], Popivanov, Schneider [24], Khe Kan Cher [20] and Popivanov, Popov [26].

Lemma 1.1. *(see [13]) Let ρ, φ and t be the polar coordinates in $\mathbb{R}^3$: $x_1=\rho\cos\varphi$, $x_2=\rho\sin\varphi$. Let us define the functions*

$$H_k^n(\rho,t):=\sum_{i=0}^{k}A_i^k\frac{t(\rho^2-t^2)^{n-3/2-k-i}}{\rho^{n-2i}} \tag{1.2}$$

and

$$E_k^n(\rho,t)=\sum_{i=0}^{k}B_i^k\frac{(\rho^2-t^2)^{n-1/2-k-i}}{\rho^{n-2i}},$$

where the coefficients are

$$A_i^k := (-1)^i \frac{(k-i+1)_i(n-1/2-k-i)_i}{i!(n-i)_i}$$

and

$$B_i^k := (-1)^i \frac{(k-i+1)_i(n+1/2-k-i)_i}{i!(n-i)_i}$$

with $(a)_i := a(a+1)...(a+i-1)$. *Then for* $n \in \mathbb{N}$, $n \geq 4$, *the functions*

$$V_k^{n,1}(\rho,t,\varphi) = H_k^n(\rho,t)\cos n\varphi \qquad \text{and} \qquad V_k^{n,2}(\rho,t,\varphi) = H_k^n(\rho,t)\sin n\varphi$$

are classical solutions of the homogeneous Problem $P1^*$ *for all* $k = 0, 1, ..., \left[\frac{n}{2}\right] - 2$, *and the functions*

$$W_k^{n,1}(\rho,t,\varphi) = E_k^n(\rho,t)\cos n\varphi \qquad \text{and} \qquad W_k^{n,2}(\rho,t,\varphi) = E_k^n(\rho,t)\sin n\varphi$$

are classical solutions of the homogeneous Problem $P2^*$ *for all* $k = 0, 1, ..., \left[\frac{n-1}{2}\right] - 1$.

A necessary condition for the existence of classical solution for Problem $P1$ (Problem $P2$) is the orthogonality of the right-hand side function f to all functions $V_k^{n,i}(\rho,t,\varphi)$ (respectively $W_k^{n,i}$). To avoid an infinite number of necessary conditions in the frame of classical solvability, we need to introduce some *generalized solutions* of Problems $P1$ and $P2$ with eventually singularity on the characteristic cone Σ_2, or only at its vertex O. Popivanov, Schneider in [24] and [25] give the following definition:

Definition 1.1. *A function* $u = u(x_1, x_2, t)$ *is called a generalized solution* of the Problem $P1$ in Ω if:

1) $u \in C^1(\bar{\Omega}\backslash O)$, $u|_{S_0\backslash O} = 0$, $u|_{S_1} = 0$;

2) the identity

$$\int_\Omega (u_t w_t - u_{x_1}w_{x_1} - u_{x_2}w_{x_2} - fw)\, dx_1 dx_2 dt = 0$$

holds for all $w \in C^1(\bar{\Omega})$, $w = 0$ on S_0, and $w = 0$ in a neighborhood of S_2.

Garabedian [10] proved the uniqueness of a classical solution of Problem $P1$. Popivanov, Schneider [25] proved the uniqueness of a *generalized solution* in $C^1(\overline{\Omega}\backslash O)$. It is known (cf. Popivanov, Schneider [25], Aldashev [1]) that for every $n \in \mathbb{N}$, $n \geq 4$, there exists a smooth function

$$f_n(x_1, x_2, t) := f_n^1(\rho, t)\cos n\varphi \in C^{n-2}(\overline{\Omega}),$$

whose corresponding *generalized solution* of Problem $P1$ near the origin O behaves like r^{1-n}, where $r = (x_1^2 + x_2^2 + t^2)^{1/2}$ is the distance to the origin. The same phenomenon appears in the case of a more general boundary condition P_α (see [11]). These singularities of the *generalized solutions* do not propagate in the direction of the bicharacteristics on the characteristic cone. It is traditionally assumed that the wave equation, whose right-hand side is sufficiently smooth in $\bar{\Omega}$, cannot have a solution with an isolated singular point. For results concerning the propagation of singularities for operators of second order we refer to Hörmander [15, Chapter 24.5]. For some related results in the case of the plane Darboux problem see [23].

Further, we study the Problem $P1$ only in the case when the right-hand side function f is a trigonometric polynomial of order l:

$$f(x_1, x_2, t) = f_0^1(\rho, t) + \sum_{n=1}^{l} \left(f_n^1(\rho, t) \cos n\varphi + f_n^2(\rho, t) \sin n\varphi \right). \tag{1.3}$$

In this case Popivanov, Schneider [25] proved the existence and uniqueness of a *generalized solution* $u(x_1, x_2, t)$ of the corresponding problem. We already know that $u(x_1, x_2, t)$ may have power type singularity at the origin. More precisely, there are solutions that have the growth of r^{1-l} at the point O. In this paper we will prove some existence and uniqueness results for Problems $P1$ and $P2$ and study the behavior of the *generalized solution* around the origin. Let us denote the weighted uniform norm

$$||f_n||_q^* := \sum_{|\alpha| \le q;\ i=1,2} \max_{0 \le t \le \rho \le 1-t} \left| r^{|\alpha|+1/2} D^\alpha f_n^i(\rho, t) \right|,$$

analogous to the weighted Sobolev norms in corner domains (see [21], [14]). Denote as usual $x_+^\alpha = x^\alpha$ for $x > 0$, and $x_+^\alpha = 0$ for $x \le 0$. Then the main results are:

Theorem 1.1. *Let us suppose that* $f(x_1, x_2, t) \in C^{(l-4)_+}(\bar{\Omega})$ *has the form* (1.3) *and that*

$$\int_\Omega V_k^{n,i}(x_1, x_2, t) f(x_1, x_2, t) dx_1 dx_2 dt = 0 \tag{1.4}$$

for all $n = 2, 3, ..., l$; $i = 1, 2$; $k = 0, 1, ..., \left[\frac{n}{2}\right] - 1$. *Then there exists an unique generalized solution* u *of Problem* $P1$. *Moreover,* $u \in C^{1+(l-4)_+}(\bar{\Omega} \backslash O)$ *and for every* $\varepsilon > 0$ *it holds the a priori estimate*

$$\begin{aligned} |u(x_1, x_2, t)| \le\ & C_1 r^{1/4} ||f_0||_0^* + C_{2,\varepsilon} r^{-\varepsilon} ||f_1||_0^* \\ & + C_3 r^{1/2} |\ln r| \sum_{k=1}^{[l/2]} ||f_{2k}||_{(2k-4)_+}^* \\ & + C_4 \sum_{k=1}^{\left[\frac{l-1}{2}\right]} ||f_{2k+1}||_{(2k-3)_+}^*, \end{aligned} \tag{1.5}$$

where $C_{2,\varepsilon}$ *depends on* ε, *but all the constants* C_1, $C_{2,\varepsilon}$, C_3 *and* C_4 *are independent on the function* f.

Theorem 1.1 gives an a priori estimate of the *generalized solution*. Now the next Theorem 1.2 provides an a priori estimate for the *generalized solution* and clarifies the significance of the above orthogonality conditions (1.4). In other worlds, for any couple (n, k) the corresponding condition "controls" one power-type singularity.

Theorem 1.2. *Let $s \in \mathbb{N}$ be such that $2 \leq s \leq l$. Suppose that $f(x_1, x_2, t) \in C^{(l-4)_+}(\overline{\Omega})$ has the form* (1.3) *and satisfies the orthogonality conditions* (1.4) *for any couple (n,k) such that $n, k \in \mathbb{N} \cup \{0\}$, $2 \leq n \leq l$, $n - 2k \geq s+1$ and $i = 1, 2$. Then there exists an unique generalized solution u of Problem $P1$ such that: $u \in C^{1+(l-4)_+}(\overline{\Omega} \backslash O)$ and the estimate*

$$|u(x_1, x_2, t)| \leq Cr^{-(s-1)} \sum_{k=0}^{l} ||f_k||^*_{(k-s-1)_+} \tag{1.6}$$

holds. If we suppose additionally that there are $m, p \in \mathbb{N} \cup \{0\}$ and $j = 1$ or 2 such that $2 \leq m \leq l$, $m - 2p = s$ and

$$\int_\Omega V_p^{m,j}(x_1, x_2, t) f(x_1, x_2, t) dx_1 dx_2 dt \neq 0, \tag{1.7}$$

then in some neighborhood of the origin one has

$$\left|u_m^j|_{S_2}(x_1, x_2)\right| \geq c(x_1^2 + x_2^2)^{-(s-1)/2}, \quad c > 0, \tag{1.8}$$

where

$$u_m^1|_{S_2}(x_1, x_2) := \int_0^{2\pi} u|_{t=|x|} \cos m\varphi \, d\varphi, \quad u_m^2|_{S_2}(x_1, x_2) := \int_0^{2\pi} u|_{t=|x|} \sin m\varphi \, d\varphi.$$

To illustrate the dependence of the singularity of the *generalized solution* on the orthogonality assumptions, let us consider the following table:

Table 1. The orthogonality conditions and the order of singularity

	l	$l-1$	$l-2$	$l-3$	...	m	...	4	3	2
1	...	...	...	...	...	...	...	$V_1^{4,i}$	◇	$V_0^{2,i}$
2	...	...	...	...	...	...	...	◇	$V_0^{3,i}$	
3	...	...	...	...	...	...	...	$V_0^{4,i}$		
...	...	...	...	...	...	...	...			
$m-2p-1$	...	...	...	...	...	$V_p^{m,i}$	...			
						...				
...	...	...	...	...	...					
$l-4$	◇	$V_1^{l-1,i}$	◇	$V_0^{l-3,i}$						
$l-3$	$V_1^{l,i}$	◇	$V_0^{l-2,i}$							
$l-2$	◇	$V_0^{l-1,i}$								
$l-1$	$V_0^{l,i}$									

Observe that both $V_k^{n,1}$ and $V_k^{n,2}$ are located in the n-th column and $(n-2k-1)$-st row of Table 1. Thus, $V_0^{n,i}$ form the right most diagonal, the next one is empty – we put in the cells "diamonds" $\diamond$, $V_1^{n,i}$ constitute the third one, and so on. The first column designates the order of singularity of the *generalized solution*.

Theorem 1.1 asserts that the *generalized solution* of Problem $P1$ is bounded if the right-hand side f is orthogonal to all the functions $V_k^{n,i}$ from the table. Theorem 1.2 specifies that (if $l \geq 2$) the singularity of the *generalized solution* is no worse than r^{1-s} if f is orthogonal to $V_k^{n,i}$ from the triangle under the $(s-1)$-st row. In other words, the functions from the k-th row of the table are "responsible" for the *generalized solutions* with behavior r^{-k} near the origin O.

The present paper is a generalization, extension and improvement of the results obtained in [26]. It consists of an introduction and five consecutive sections. Section 2 is devoted to the solutions of the homogeneous adjoint Problems $P1^*$ and $P2^*$. In Section 3 are formulated the $2-D$ boundary Problems $P12$ and $P13$, shortly related to the $3-D$ Problem $P1$. The main technical results are established in Sections 4 and 5 – we study the behavior of solutions of $2-D$ Problems $P12$ and $P13$. In the last Section 6 we give proofs of Theorem 1.1 and Theorem 1.2 based on the results of the previous two sections.

2. PROPERTIES OF THE SOLUTIONS H_K^N AND E_K^N

First, we will present three different ways to introduce the solutions of the homogeneous adjoint Problems $P1^*$ and $P2^*$. The functions H_k^n and E_k^n could be found in Khe [20] in the form

$$t\rho^{n-2k-3}(1-t^2/\rho^2)^{n-2k-3/2}F(n-k,-k;3/2;t^2/\rho^2)$$

and

$$\rho^{n-2k-1}(1-t^2/\rho^2)^{n-2k-1/2}F(n-k,-k;1/2;t^2/\rho^2),$$

where F is the hypergeometric Gaus function.

On the other hand, one could obtain $H_k^n(\rho,t)$ and $E_k^n(\rho,t)$ by differentiation of $E_0^n(\rho,t)$ with respect to t.

Lemma 2.1. (*see* [13, Theorem 4.2]) *The functions $H_k^n(\rho,t)$ and $E_k^n(\rho,t)$, defined in Lemma* 1.1, *satisfy*

$$\frac{\partial}{\partial t}H_k^n(\rho,t) = 2(n-k-1)E_{k+1}^n(\rho,t),$$

$$\frac{\partial}{\partial t}E_k^n(\rho,t) = 2(k-n+1/2)H_k^n(\rho,t)$$

and they represent some derivative of $E_0^n(\rho,t)$ over t:

$$H_k^n(\rho,t) = \frac{(-1)^{k+1}}{(2n-2k-1)_{2k+1}}\left(\frac{\partial}{\partial t}\right)^{2k+1}\left(\frac{(\rho^2-t^2)^{n-1/2}}{\rho^n}\right),$$

$$E_k^n(\rho,t) = \frac{(-1)^k}{(2n-2k)_{2k}} \left(\frac{\partial}{\partial t}\right)^{2k} \left(\frac{(\rho^2-t^2)^{n-1/2}}{\rho^n}\right).$$

Remark 2.1. This procedure of differentiating the function E_0^n (or $W_0^{n,i}$) with respect to t will (as in Lemma 1.1) produce solutions $V_k^{n,i}$ and $W_k^{n,i}$ of the equation (1.1), but for $k \geq n/2$ the smoothness at the point O will be lost.

Remark 2.2. The solutions of the adjoint Problem $P2^*$ ($P1^*$) given in Lemma 1.1 are not orthogonal. For example, one could check this out for $W_0^{n,i}$ and $W_1^{n,i}$. It is sufficient to show that

$$K := \int_0^{1/2}\int_t^{1-t} E_0^n(\rho,t)E_1^n(\rho,t)\rho d\rho dt \neq 0.$$

In fact, Lemma 2.1 implies $\partial^2 E_0^n/\partial t^2(\rho,t) = cE_1^n(\rho,t)$ for some constant c and therefore

$$\begin{aligned} cK &= \int_0^{1/2}\int_t^{1-t} E_0^n(\rho,t)\frac{\partial^2 E_0^n}{\partial t^2}(\rho,t)\rho\, d\rho dt \\ &= \int_{1/2}^{1} E_0^n(\rho,1-\rho)\frac{\partial E_0^n}{\partial t}(\rho,1-\rho)\,\rho\, d\rho - \int_0^{1/2}\int_t^{1-t}\left(\frac{\partial E_0^n}{\partial t}(\rho,t)\right)^2 \rho\, d\rho dt < 0, \end{aligned}$$

because $E_0^n(\rho,t) \geq 0$ and $\partial E_0^n/\partial t(\rho,t) \leq 0$ for $t \leq \rho$.

Remark 2.3. The functions $H_k^n(\rho,t)$ and $E_k^n(\rho,t)$ are linearly independent. Indeed, suppose that some linear combination of these functions is zero. Then from Lemma 2.1 it follows that E_0^n as a function of t is a solution (for a fixed ρ) of a homogeneous linear differential equation with constant coefficients. Therefore E_0^n must be a finite sum of quasi-polynomials of t, which obviously is not true.

A basic tool for our treatment of Protter problems are the Legendre functions P_ν (see (3.8) below). Some properties of the Legendre functions P_ν one can find in [9]. The next lemma plays a key role in the last section.

Lemma 2.2. *Let us denote*

$$h_k^\nu(\xi,\eta) = \int_\eta^\xi s^k P_\nu\left(\frac{\xi\eta+s^2}{s(\xi+\eta)}\right) ds.$$

If $\nu = n - 1/2$, *then it hold:* (**a**) *for* $i = 0, 1, ..., \left[\dfrac{\nu-1}{2}\right]$

$$h_{\nu-2i-2}^\nu(\xi,\eta)\Big|_{\xi=\frac{\rho+t}{2};\eta=\frac{\rho-t}{2}} = c_i^n \rho^{1/2} H_i^n(\rho,t)$$

and (**b**) *for* $i = 0, 1, ..., \left[\frac{\nu}{2}\right]$

$$\left(\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\eta}\right) h^{\nu}_{\nu-2i}(\xi,\eta) \left.\right|_{\xi=(\rho+t)/2;\eta=(\rho-t)/2} = d^n_i \rho^{1/2} E^n_i(\rho, t),$$

where the constants c^n_i, $d^n_i \neq 0$.

Proof. (**a**) We will calculate the integrals $h^{\nu}_k(\xi,\eta)$ using the Mellin transform, given by

$$f^*(s) = \int_0^{\infty} t^{s-1} f(t)\, dt,$$

and the Mellin convolution "$\circ$", defined by

$$(f \circ g)(x) = \int_0^{\infty} f\left(\frac{x}{t}\right) g(t) \frac{dt}{t}.$$

Recall that the relation between both is (see [22, formula (1.2)])

$$(f \circ g)^*(s) = f^*(s) g^*(s). \tag{2.1}$$

To apply the last formula to $h^{\nu}_k(\xi,\eta)$, let us introduce new variables x, y and z defined by

$$2\sqrt{x} = \sqrt{\frac{\xi}{\eta}} + \sqrt{\frac{\eta}{\xi}}; \quad 2\sqrt{y} = \frac{\sqrt{\xi\eta}}{s} + \frac{s}{\sqrt{\eta\xi}}; \quad z = \sqrt{\xi\eta}.$$

Then we have

$$\frac{\xi\eta + s^2}{s(\xi+\eta)} = \sqrt{\frac{y}{x}}\,; \quad \frac{s}{\sqrt{\eta\xi}} = \sqrt{y} \pm \sqrt{y-1};$$

$$\frac{d(2\sqrt{y})}{ds} = \frac{d}{ds}\left(\frac{\sqrt{\xi\eta}}{s} + \frac{s}{\sqrt{\eta\xi}}\right) = \frac{s^2 - z^2}{s^2 z};$$

$$(\sqrt{y} + \sqrt{y-1})^2 - 1 = 2\sqrt{y-1}(\sqrt{y-1} + \sqrt{y});$$

$$(\sqrt{y} - \sqrt{y-1})^2 - 1 = 2\sqrt{y-1}(\sqrt{y-1} - \sqrt{y});$$

when $s = \sqrt{\xi\eta}$, we have $y = 1$, and $y = x$ for $s = \xi$ or η . Substituting in h^{ν}_k, we find

$$h^{\nu}_k(x, z) = \frac{z^{k+1}}{2} \int_0^{\infty} \left(\frac{x}{y} - 1\right)^0_+ P_{\nu}\left(\sqrt{\frac{y}{x}}\right) (y-1)^{-1/2}_+ \left((\sqrt{y} + \sqrt{y-1})^{k+1}\right.$$
$$\left. + (\sqrt{y} - \sqrt{y-1})^{k+1}\right) \sqrt{y} \frac{dy}{y}.$$

Now we are ready to use (2.1) and formulae (11.13(4)), (2.10(4)) from [22] – applying the Mellin transform over x (here "$\mapsto$" means "transforms into"):

$$(x-1)^0_+ P_{\nu}\left(\sqrt{\frac{1}{x}}\right) \mapsto \frac{\Gamma(-s)\Gamma\left(\frac{1}{2} - s\right)}{\Gamma\left(\frac{1-\nu}{2} - s\right)\Gamma\left(1 + \frac{\nu}{2} - s\right)};$$

$$\sqrt{y}(y-1)_+^{-1/2}\left((\sqrt{y}+\sqrt{y-1})^{k+1} \quad + \quad (\sqrt{y}-\sqrt{y-1})^{k+1}\right)$$

$$\mapsto 2\pi\frac{\Gamma\left(\frac{1+k+1}{2}-s-\frac{1}{2}\right)\Gamma\left(\frac{1-k-1}{2}-s-\frac{1}{2}\right)}{\Gamma\left(1-s-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-s-\frac{1}{2}\right)}$$

and therefore

$$h_k^\nu(x,z)z^{-k-1} \mapsto \pi\frac{\Gamma\left(\frac{1+k}{2}-s\right)\Gamma\left(-\frac{1+k}{2}-s\right)}{\Gamma\left(\frac{1-\nu}{2}-s\right)\Gamma\left(1+\frac{\nu}{2}-s\right)}. \tag{2.2}$$

To find the inverse image of the right-hand side, denote as usual by I_{0+}^α the Riemann-Liouville fractional integral (derivative) of order α:

$$I_{0+}^\alpha f(x) := \frac{1}{\Gamma(\alpha)}\int_0^x f(t)(x-t)^{\alpha-1}dt,\ \alpha>0;$$

$$I_{0+}^0 f(x) := f(x);$$

$$I_{0+}^{-\alpha} f(x) := \left(I_{0+}^\alpha\right)^{-1} f(x) := \left(\frac{d}{dx}\right)^{[\alpha]+1}\left(I_{0+}^{1-\{\alpha\}}f\right)(x) \ ,\ \alpha>0.$$

Under these notations, the right-hand side of (2.2) is the Mellin transform of the function

$$\frac{\pi}{\Gamma\left(\frac{\nu-k+1}{2}\right)}x^{\frac{\nu+1}{2}}I_{0+}^{\frac{k-\nu+2}{2}}\left(x^{-\frac{k+\nu+3}{2}}(x-1)_+^{\frac{\nu-k-1}{2}}\right).$$

Indeed, using (2.2.(4)), (1.10) and (1.4) from [22], for $k<\nu-1$ we find

$$x^{\frac{\nu+1}{2}}I_{0+}^{\frac{k-\nu+2}{2}}\left(x^{-\frac{k+\nu+3}{2}}(x-1)_+^{\frac{\nu-k-1}{2}}\right) \mapsto \Gamma\left(\frac{\nu-k+1}{2}\right)\frac{\Gamma\left(-\frac{1+k}{2}-s\right)\Gamma\left(\frac{1+k}{2}-s\right)}{\Gamma\left(\frac{1-\nu}{2}-s\right)\Gamma\left(1+\frac{\nu}{2}-s\right)}.$$

Hence, taking $k=\nu-2i-2$, for some constants $C_{\nu,i}$ we have

$$h_{\nu-2i-2}^\nu(x,z)z^{-\nu+2i+1} = C_{\nu,i}x^{\frac{\nu+1}{2}}\left(\frac{d}{dx}\right)^i\left(x^{-\nu+i-1/2}(x-1)_+^{i+1/2}\right).$$

Let us now return to the variables ρ and t :

$$\begin{aligned}
z^{\nu-2i-1}x^{\frac{\nu+1}{2}} \quad & (d/dx)^i\left(x^{-\nu+i-1/2}(x-1)^{i+1/2}\right)\\
&= (\xi\eta)^{\frac{\nu-2i-1}{2}}\sum_{j=0}^{i} d_j'\left(\frac{(\xi+\eta)^2}{\xi\eta}\right)^{-\nu/2+j}\left(\frac{(\xi-\eta)^2}{\xi\eta}\right)^{i+1/2-j}\\
&= (\rho^2-t^2)^{\frac{\nu-2i-1}{2}}\sum_j d_j''\left(\frac{\rho^2}{(\rho^2-t^2)}\right)^{-\nu/2+j}\left(\frac{t^2}{(\rho^2-t^2)}\right)^{i+1/2-j}\\
&= \sum_{j+l\le i} d_{j,l}t(\rho^2-t^2)^{\nu-i-1-(j+l)}\rho^{-\nu+2(j+l)}\\
&= \sum_{j=0}^{i} a_j^i\frac{t(\rho^2-t^2)^{n-3/2-j-i}}{\rho^{n-2j-1/2}}.
\end{aligned}$$

In order to determine the coefficients a_j^i, one could notice that from the definition of h_k^ν the function

$$\rho^{-1/2} h^\nu_{\nu-2i-2}\left(\frac{\rho+t}{2}, \frac{\rho-t}{2}\right) \sin n\varphi$$

satisfies the wave equation. Therefore, after substituting in the equation, we find $a_j^i = a_0^i A_j^i$ and

$$\rho^{-1/2} h^\nu_{\nu-2i-2}\left(\frac{\rho+t}{2}, \frac{\rho-t}{2}\right) = a_i^\nu H_i^n(\rho, t).$$

(b) Let us find the functions $(\partial/\partial\xi - \partial/\partial\eta)\, h^\nu_{\nu-2i}(\xi,\eta)$, where $i = 0, 1, ..., \left[\frac{\nu}{2}\right]$. Notice that for $i \geq 1$, due to Lemma 2.1 and **(a)**,

$$\rho^{-1/2}\left(\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\eta}\right) h^\nu_{\nu-2i}(\xi,\eta) = C_i^n \frac{\partial}{\partial t} H^n_{i-1}(\rho,t) = C_i^{n\prime} E_i^n(\rho,t).$$

Only the case $i = 0$ has been omitted, i.e. we need to calculate

$$\left(\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\eta}\right) h^\nu_\nu(\xi,\eta).$$

Therefore we consider the function

$$\frac{\partial}{\partial t} z^{k+1} x^{\frac{\nu+1}{2}} I_{0+}^{\frac{k-\nu+2}{2}}\left(x^{-\frac{k+\nu+3}{2}}(x-1)_+^{\frac{\nu-k-1}{2}}\right),$$

where $k = \nu$ and

$$x = \frac{\rho^2}{\rho^2 - t^2}; \quad 2z = (\rho^2 - t^2)^{1/2}.$$

That is

$$\frac{\partial}{\partial t} z^{\nu+1} x^{\frac{\nu+1}{2}} I^1_{0+}\left(x^{-\nu-\frac{3}{2}}(x-1)_+^{-1/2}\right) = \frac{\partial}{\partial t} z^{\nu+1} x^{\frac{\nu+1}{2}} \int_0^x \tau^{-\nu-\frac{3}{2}}(\tau-1)_+^{-1/2} d\tau$$

$$= \frac{\partial}{\partial t}\rho^{\nu+1} 2^{-\nu-1} \int_1^x \tau^{-\nu-\frac{3}{2}}(\tau-1)^{-1/2} d\tau = 2^{-\nu-1}\rho^{\nu+1} x^{-\nu-\frac{3}{2}}(x-1)^{-1/2}\frac{\partial x}{\partial t}$$

$$= 2^{-\nu-1}\rho^{\nu+1}\frac{(\rho^2-t^2)^{\nu+3/2}}{\rho^{2\nu+3}} \frac{(\rho^2-t^2)^{1/2}}{t} \frac{2t\rho^2}{(\rho^2-t^2)^2} = 2^{-\nu}\frac{(\rho^2-t^2)^\nu}{\rho^\nu}.$$

Hence for $\nu = n - 1/2$ we conclude that

$$\rho^{-1/2}\left(\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\eta}\right) h^\nu_\nu(\xi,\eta) = \rho^{-1/2} C_n \frac{(\rho^2-t^2)^{n-1/2}}{\rho^{n-1/2}} = C_n E_0^n(\rho,t). \qquad \square$$

3. SOLVING PROBLEM P1

In terms of Theorems 1.1 and 1.2 it is sufficient to study the Problem $P1$ only when the right-hand side f of the wave equation is simply

$$f(\rho,t,\varphi) = f_n^1(\rho,t)\cos n\varphi + f_n^2(\rho,t)\sin n\varphi, \quad n \in \mathbb{N}\cup\{0\}.$$

Then we seek solutions of the wave equation of the same form:

$$u(\rho, t, \varphi) = u_n^1(\rho, t)\cos n\varphi + u_n^2(\rho, t)\sin n\varphi.$$

Thus Problem P1 reduces to the following one:

Problem P12. To solve the equation

$$(u_n)_{\rho\rho} + \frac{1}{\rho}(u_n)_\rho - (u_n)_{tt} - \frac{n^2}{\rho^2}u_n = f_n \tag{3.1}$$

in $\Omega_0 = \{0 < t < 1/2;\ t < \rho < 1 - t\} \subset \mathbb{R}^2$ with the boundary conditions

P12 $\quad u_n(\rho, 0) = 0$ for $0 < \rho \leq 1$ and $u_n(\rho, 1-\rho) = 0$ for $1/2 \leq \rho \leq 1$.

Let us now introduce new coordinates

$$\xi = \frac{\rho + t}{2}\ ; \quad \eta = \frac{\rho - t}{2}, \tag{3.2}$$

and set

$$v(\xi, \eta) = \rho^{1/2}u_n(\rho, t)\ ; \quad g(\xi, \eta) = \rho^{1/2}f_n(\rho, t). \tag{3.3}$$

Denoting $\nu = n - \dfrac{1}{2}$, one transforms Problem $P12$ into

Problem P13. To find a solution $v(\xi, \eta)$ of the equation

$$v_{\xi\eta} - \frac{\nu(\nu+1)}{(\xi+\eta)^2}v = g \tag{3.4}$$

in the domain $D = \{0 < \xi < 1/2;\ 0 < \eta < \xi\}$ with the following boundary conditions:

P13 $\quad v(\xi, \xi) = 0$ for $\xi \in (0, 1/2)$ and $v(1/2, \eta) = 0$ for $\eta \in (0, 1/2)$.

Problems $P12$ and $P13$ have been introduced in [25], although the change of coordinates $\xi = 1 - \rho - t$ and $\eta = 1 - \rho + t$ is used there instead of (3.2). Of course, because the solution of Problem $P1$ may be singular, the same is true for the solutions of $P12$ and $P13$. For that reason, Popivanov and Schneider [25] have defined and proved the existence and uniqueness of generalized solutions of Problems $P12$ and $P13$, which correspond to the *generalized solution* of Problem $P1$. Further, by "solution" of Problem $P12$ or $P13$ we will mean exactly this unique generalized solution.

Remark 3.1. Notice that even when the right-hand side function $f_n(\rho, t)$ belongs to $C^k(\overline{\Omega}_0)$, the corresponding function $g(\xi, \eta) = \rho^{1/2}f_n(\rho, t)$ in (3.4) belongs to $C^k(\overline{\Omega}_0 \backslash O)$, but its derivative may not be continuous at the origin O. At the same time, when the solution $v(\xi, \eta)$ of Problem $P13$ is bounded, the solution $u_n(\rho, t) = \rho^{-1/2}v(\xi, \eta)$ of Problem $P12$ may be singular.

Nevertheless, we will solve Problem $P13$ instead of Problem $P1$. We can construct the solution of the Problem $P13$ using two different methods. First, following Popivanov, Schneider [25], one could use the equivalent integral equation

$$U(\xi_0, \eta_0) = \int\limits_{\xi=\xi_0}^{\xi=\frac{1}{2}} \int\limits_{\eta=\eta_0}^{\eta=\xi_0} \left(\frac{\nu(\nu+1)}{(\xi+\eta)^2} U(\xi, \eta) + g(\xi, \eta) \right) d\eta d\xi \tag{3.5}$$

and construct the solution as a limit of a sequence of successive approximations $U^{(k)}$ defined by

$$U^{(0)}(\xi_0, \eta_0) = \int\limits_{\xi=\xi_0}^{\xi=\frac{1}{2}} \int\limits_{\eta=\eta_0}^{\eta=\xi_0} g(\xi, \eta) d\eta d\xi, \tag{3.6}$$

$$U^{(k+1)}(\xi_0, \eta_0) = \int\limits_{\xi=\xi_0}^{\xi=\frac{1}{2}} \int\limits_{\eta=\eta_0}^{\eta=\xi_0} \left(\frac{\nu(\nu+1)}{(\xi+\eta)^2} U^{(k)}(\xi, \eta) + g(\xi, \eta) \right) d\eta d\xi. \tag{3.7}$$

On the other hand, notice that the function

$$R(\xi_1, \eta_1, \xi, \eta) = P_\nu \left(\frac{(\xi - \eta)(\xi_1 - \eta_1) + 2\xi_1\eta_1 + 2\xi\eta}{(\xi_1 + \eta_1)(\xi + \eta)} \right) \tag{3.8}$$

is a Riemann function for the equation (3.4) (Copson [7]). Therefore, we can construct the function v as a solution of a Goursat's problem in D with boundary conditions $v(1/2, \eta) = 0$ and $v(\xi, 0) = \varphi(\xi)$ with some unknown function $\varphi(\xi)$, which will be determined later:

$$v(\xi, \eta) = \varphi(\xi) + \int\limits_{\xi}^{\frac{1}{2}} \varphi(\xi_1) \frac{\partial}{\partial \xi_1} R(\xi_1, 0, \xi, \eta) d\xi_1 - \int\limits_{\xi}^{\frac{1}{2}} \int\limits_{0}^{\eta} R(\xi_1, \eta_1, \xi, \eta) g(\xi_1, \eta_1) d\eta_1 d\xi_1. \tag{3.9}$$

Now, following Aldashev [1], the boundary condition $v(\xi, \xi) = 0$ gives the equation

$$G(\xi) = \varphi(\xi) + \int\limits_{\xi}^{\frac{1}{2}} \varphi(\xi_1) \frac{\partial}{\partial \xi_1} P_\nu \left(\frac{\xi}{\xi_1} \right) d\xi_1 = - \int\limits_{\xi}^{\frac{1}{2}} \varphi'(\xi_1) P_\nu \left(\frac{\xi}{\xi_1} \right) d\xi_1 \tag{3.10}$$

for the function $\varphi(\xi)$, where

$$G(\xi) = \int\limits_{\xi}^{\frac{1}{2}} \int\limits_{0}^{\xi} P_\nu \left(\frac{\xi_1\eta_1 + \xi^2}{\xi(\xi_1 + \eta_1)} \right) g(\xi_1, \eta_1) d\eta_1 d\xi_1. \tag{3.11}$$

According to formulae (35.17), (35.28) from [28], the integral equation (3.10) is invertible and we have

$$\frac{d}{d\xi}\varphi(\xi) = -\xi \frac{d^2}{d\xi^2} \xi \int\limits_{\xi}^{\frac{1}{2}} P_\nu \left(\frac{\xi_1}{\xi} \right) \frac{G(\xi_1)}{\xi_1^2} d\xi_1 = G'(\xi) + \frac{d}{d\xi} \int\limits_{\xi}^{\frac{1}{2}} P'_\nu \left(\frac{\xi_1}{\xi} \right) \frac{G(\xi_1)}{\xi_1} d\xi_1.$$

Finally, due to $\varphi(1/2) = G(1/2) = 0$, it follows

$$\varphi(\xi) = G(\xi) + \int\limits_{\xi}^{\frac{1}{2}} P'_\nu \left(\frac{\xi_1}{\xi} \right) \frac{G(\xi_1)}{\xi_1} d\xi_1. \qquad \square \tag{3.12}$$

Using the sequence $\{U^{(k)}\}$ defined by (3.6), (3.7), we have the following

Lemma 4.1. *Let $0 < \varepsilon < 1$ and suppose that $|\nu(\nu+1)| < \varepsilon(1+\varepsilon)$. Then the solution $U(\xi,\eta)$ of the integral equation (3.5) in D satisfies the estimate*

$$|U(\xi,\eta)| \leq C(\xi-\eta)(\xi+\eta)^{-\varepsilon} \sup_D |g|,$$

where the constant C depends only on ν and ε.

Proof. The key point in the proof is the estimate for the integrals:

$$\begin{aligned}
I_\varepsilon &:= \int\limits_{\xi=\xi_0}^{\xi=\frac{1}{2}} \int\limits_{\eta=\eta_0}^{\eta=\xi_0} (\xi-\eta)(\xi+\eta)^{-\varepsilon-2} d\eta d\xi \\
&= \int\limits_{\xi=\xi_0}^{\xi=\frac{1}{2}} \int\limits_{\eta=\eta_0}^{\eta=\xi_0} \left((\xi+\eta)^{-\varepsilon-1} - 2\eta(\xi+\eta)^{-\varepsilon-2}\right) d\eta d\xi \\
&= \int\limits_{\eta_0}^{\xi_0} \left(\frac{1}{\varepsilon}(\xi_0+\eta)^{-\varepsilon} - \frac{1}{\varepsilon}(\frac{1}{2}+\eta)^{-\varepsilon} - \frac{2\eta}{1+\varepsilon}(\xi_0+\eta)^{-\varepsilon-1} + \frac{2\eta}{1+\varepsilon}(\frac{1}{2}+\eta)^{-\varepsilon-1}\right) d\eta \\
&\leq \int\limits_{\eta_0}^{\xi_0} \left(\frac{1}{\varepsilon}(\xi_0+\eta)^{-\varepsilon} - \frac{2\eta}{1+\varepsilon}(\xi_0+\eta)^{-\varepsilon-1}\right) d\eta \\
&= \int\limits_{\eta_0}^{\xi_0} \left(\frac{1-\varepsilon}{\varepsilon(1+\varepsilon)}(\xi_0+\eta)^{-\varepsilon} + \frac{2\xi_0}{1+\varepsilon}(\xi_0+\eta)^{-\varepsilon-1}\right) d\eta \\
&= -\frac{1}{\varepsilon(1+\varepsilon)}(\xi_0+\eta_0)^{1-\varepsilon} + \frac{2\xi_0}{\varepsilon(1+\varepsilon)}(\xi_0+\eta_0)^{-\varepsilon} \\
&= \frac{1}{\varepsilon(1+\varepsilon)}(\xi_0-\eta_0)(\xi_0+\eta_0)^{-\varepsilon}.
\end{aligned}$$

For the function $U^{(0)}$ we find

$$\sup_\Omega |U^{(0)}| \leq \frac{1}{2}(\xi_0-\eta_0) \sup_D |g| \leq (\xi_0-\eta_0)(\xi_0+\eta_0)^{-\varepsilon} \sup_D |g|.$$

Now, because of

$$\left|(U^{(k+1)} - U^{(k)})(\xi_0,\eta_0)\right| \leq \int\limits_{\xi=\xi_0}^{\xi=\frac{1}{2}} \int\limits_{\eta=\eta_0}^{\eta=\xi_0} \frac{|\nu(\nu+1)|}{(\xi+\eta)^2} \left|(U^{(k)} - U^{(k-1)})(\xi,\eta)\right| d\eta d\xi,$$

if take $\alpha := \dfrac{|\nu(\nu+1)|}{\varepsilon(1+\varepsilon)}$, we have by induction

$$\left|(U^{(k)} - U^{(k-1)})(\xi_0,\eta_0)\right| \leq \alpha^k (\xi_0-\eta_0)(\xi_0+\eta_0)^{-\varepsilon} \sup_D |g|.$$

Indeed, using the calculations of I_ε, we find that

$$\left|(U^{(k+1)} - U^{(k)})(\xi_0, \eta_0)\right| \leq \int\limits_{\xi=\xi_0}^{\xi=\frac{1}{2}} \int\limits_{\eta=\eta_0}^{\eta=\xi_0} \frac{|\nu(\nu+1)|}{(\xi+\eta)^{2+\varepsilon}} \alpha^k (\xi - \eta) \sup_D |g| d\eta d\xi \tag{4.1}$$

$$\leq \alpha^{k+1}(\xi_0 - \eta_0)(\xi_0 + \eta_0)^{-\varepsilon} \sup_D |g|.$$

Finally, we arrive at the estimate

$$\begin{aligned} |U^{(k+1)}(\xi_0, \eta_0)| \leq & \sum_{i=0}^{k} \left|(U^{(i+1)} - U^{(i)})(\xi_0, \eta_0)\right| \\ & \leq \sum_{i=0}^{k} \alpha^{i+1}(\xi_0 - \eta_0)(\xi_0 + \eta_0)^{-\varepsilon} \sup_D |g| \\ & = \frac{(1-\alpha^{k+2})}{(1-\alpha)}(\xi_0 - \eta_0)(\xi_0 + \eta_0)^{-\varepsilon} \sup_D |g|, \end{aligned}$$

which, together with (4.1), shows that if $\alpha < 1$, the sequence $\{U^{(k)}\}$ uniformly converges to a solution $U(\xi, \eta)$ of

$$U(\xi_0, \eta_0) = \int\limits_{\xi=\xi_0}^{\xi=\frac{1}{2}} \int\limits_{\eta=\eta_0}^{\eta=\xi_0} \left(\frac{\nu(\nu+1)}{(\xi+\eta)^2} U(\xi, \eta) + g(\xi, \eta) \right) d\eta d\xi.$$

Even more, it holds the estimate

$$|U(\xi, \eta)| \leq \frac{1}{(1-\alpha)}(\xi - \eta)(\xi + \eta)^{-\varepsilon} \sup_D |g|. \qquad \square$$

Now, for $n = 0$ or 1 we have $\nu = -1/2$ or $1/2$, respectively, and we will apply Lemma 4.1 with suitable ε.

Theorem 4.1. *For the solution $u_n(\rho, t)$ of Problem P12 with right-hand side function $f_n \in C(\overline{\Omega}_0)$ the following a priori estimate holds:*

(i) *for the case $n = 0$*

$$|u_0(\rho, t)| \leq C\rho^{1/4} \max_{\overline{\Omega}_0} |\rho^{1/2} f_0|$$

with constant C, independent of the function $f_0(\rho, t)$;

(ii) *if $n = 1$, then for every $\delta > 0$ there exists a constant C_δ, independent of the function $f_1(\rho, t)$, such that*

$$|u_1(\rho, t)| \leq C_\delta \rho^{-\delta} \max_{\overline{\Omega}_0} |\rho^{1/2} f_1|.$$

Proof. Notice that when $n = 0$ and $n = 1$ we have $|\nu(\nu+1)| = 1/4$ and $|\nu(\nu+1)| = 3/4$, respectively. Therefore one could apply Lemma 4.1 for the solution

$v(\xi,\eta) = U(\xi,\eta)$ of Problem $P13$ with $\varepsilon = 1/4$ and $\varepsilon = 1/2+\delta$, respectively. The assertion follows from the relation (3.3):

$$|u(\rho,t)| = \left|\rho^{-1/2} v\left(\frac{\rho+t}{2}, \frac{\rho-t}{2}\right)\right| \le C'\rho^{1-\varepsilon-1/2}\max_{\overline{D}}|g| \le C\rho^{1/2-\varepsilon}\max_{\overline{\Omega}_0}|\rho^{1/2}f|. \square$$

5. THE CASE $n \ge 2$

In this case, unlike the above approach, the behavior of the solution is studied in [26], using the properties of the Riemann function (3.8), given by Legendre functions P_ν. Let us remind some of these results here. For the function $v(\xi,\eta)$, defined by (3.9), (3.12) and (3.11), it is not hard to see that $v(\xi,0) = \varphi(\xi)$ may blow up when ξ tends to 0. Nevertheless, one could control the growth of

$$I(\xi) := \int_{\xi}^{\frac{1}{2}} P'_\nu\left(\frac{\xi_1}{\xi}\right)\frac{G(\xi_1)}{\xi_1}d\xi_1 \tag{5.1}$$

with the help of the following lemma:

Lemma 5.1. (*see* [26, Lemma 3.1]) *Let $\nu > 1$ be a real number with integral part $[\nu]$ and fractional part $\{\nu\} = \nu - [\nu] \neq 0$. Suppose that $G \in C^{[\nu-2]_+}(0,1/2]$, $|G^{(k)}(\xi)| \le A\xi^{1-k}$ for $k = 0,1,...,[\nu-2]_+$ for some constant A, and*

$$\int_0^{\frac{1}{2}} \xi^{\nu-2i-2}G(\xi)d\xi = 0 \quad \text{for } \ i = 0,1,...,\left[\frac{\nu-1}{2}\right]. \tag{5.2}$$

Then the function $I(\xi)$, defined by (5.1), *is $C^{[\nu]-1}(0,1/2]$. More precisely, there is a constant C, independent of $G(\xi)$, such that:*
(i) $|I(\xi)| \le CA\xi$ *if $[\nu]$ is an odd number;*
(ii) $|I(\xi)| \le CA\xi^{1-\{\nu\}}$ *if $[\nu]$ is an even number.*

Besides, Lemma 3.2 from [26] asserts that each of the orthogonality conditions (5.2) actually "controls" one power-type singularity of the function $I(\xi)$:

Lemma 5.2. *Let $\nu > 1, \{\nu\} \neq 0$, and p be a nonnegative integer, $p \le [(\nu-1)/2]$. Suppose that $G \in C^{(2p-1)_+}(0,1/2]$, $|G^{(k)}(\xi)| \le A\xi^{2p-[\nu]+1-k}$ for $k = 0,1,...,(2p-1)_+$ for some constant A, and*

$$\int_0^{\frac{1}{2}} \xi^{\nu-2i-2}G(\xi)d\xi = 0 \qquad \text{for } i = 0,1,...,p-1. \tag{5.3}$$

Then the estimate

$$|I(\xi)| \le C_1 A\xi^{-(\nu-2p-1)}$$

holds for some constant C_1, independent of $G(\xi)$. Moreover, if

$$\int_0^{\frac{1}{2}} \xi^{\nu-2p-2} G(\xi) d\xi \neq 0, \tag{5.4}$$

then

$$|I(\xi)| \geq C_2 \xi^{-(\nu-2p-1)}$$

for $C_2 > 0$ and sufficiently small ξ.

In other words, one has to impose some conditions on the right-hand side g of Problem $P13$ to secure certain behavior of the solution v. In fact, the definition (3.11) of the function $G(\xi)$ gives the equality (see [26])

$$\int_0^{\frac{1}{2}} \xi^{\nu-2i-2} G(\xi) d\xi = \int_0^{\frac{1}{2}} \int_0^{\xi_1} \left(\int_{\eta_1}^{\xi_1} \xi^{\nu-2i-2} P_\nu \left(\frac{\xi_1 \eta_1 + \xi^2}{\xi(\xi_1 + \eta_1)} \right) d\xi \right) g(\xi_1, \eta_1) d\eta_1 d\xi_1 .$$

It shows that one needs orthogonality of g to the functions $h^\nu_{\nu-2i-2}$ defined in Lemma 2.2. As a result we are able to prove the following

Theorem 5.1. (*see* [26, Theorem 4.1]) *Let $n \in \mathbb{N}$, $n \geq 2$. Suppose that $g \in C^{(n-4)_+}(\overline{D}\backslash O)$, $|D^\alpha g(\xi, \eta)| \leq A\xi^{-|\alpha|}$ for $|\alpha| \leq (n-4)_+$, and the orthogonality conditions*

$$\int_0^{\frac{1}{2}} \int_0^{\xi} h^\nu_{\nu-2i-2}(\xi, \eta) g(\xi, \eta) d\eta d\xi = 0 \quad \text{for } i = 0, 1, ..., \left[\frac{\nu - 1}{2} \right] \tag{5.5}$$

are satisfied with $\nu = n - 1/2$. Then the solution $v(\xi, \eta)$ of Problem $P13$ belongs to $C^{(n-4)_+ + 1}(\overline{D}\backslash O)$ and satisfies the following estimates:

(i) *if n is an even number, then*

$$|v(\xi, \eta)| \leq CA\xi |\ln \xi|;$$

(ii) *if n is an odd number, then*

$$|v(\xi, \eta)| \leq CA\xi^{1/2}.$$

In both cases the constant C does not depend on the function $g(\xi, \eta)$.

In the same way one gets the following theorem, which corresponds to Lemma 5.2:

Theorem 5.2. (*see* [26, Theorem 4.2]) *Let p be a nonnegative integer and $p \leq [(\nu-1)/2]$. Suppose that the function $g \in C^{(2p-2)_+}(\overline{D}\backslash O)$, $|D^\alpha g(\xi, \eta)| \leq A\xi^{-|\alpha|}$ for $|\alpha| \leq (2p-2)_+$ and*

$$\int_0^{\frac{1}{2}} \int_0^{\xi} h^\nu_{\nu-2i-2}(\xi, \eta) g(\xi, \eta) d\eta d\xi = 0 \quad \text{for } i = 0, 1, ..., p-1. \tag{5.6}$$

Then the estimate

$$|v(\xi,\eta)| \le CA\xi^{-(\nu-2p-1)} \tag{5.7}$$

holds for some constant C, independent of the function $g(\xi,\eta)$. Moreover, the condition

$$\int_0^{\frac{1}{2}}\int_0^{\xi} h^{\nu}_{\nu-2p-2}(\xi,\eta)g(\xi,\eta)d\eta d\xi \neq 0 \tag{5.8}$$

implies that the lower estimate

$$|v(\xi,0)| \ge c\xi^{-(\nu-2p-1)} \tag{5.9}$$

holds for some constant $c > 0$ and sufficiently small ξ.

All these preparations and Lemma 2.2 lead us to the following estimate for the solution of Problem $P12$:

Theorem 5.3. *Suppose that $n \in \mathbb{N}$, $n \ge 2$, $f_n(\rho,t) \in C^{(n-4)_+}(\overline{\Omega}_0\backslash O)$, and there is a constant A such that $|\rho^{|\alpha|+1/2}D^{\alpha}f_n(\rho,t)| \le A$ for $|\alpha| \le (n-4)_+$. Let also there hold the orthogonality conditions*

$$\int_0^{\frac{1}{2}}\int_t^{1-t} H^n_i(\rho,t)f_n(\rho,t)\rho d\rho dt = 0 \quad \textit{for } i = 0,1,...,\left[\frac{n}{2}\right]-1. \tag{5.10}$$

Then the solution $u_n(\rho,t)$ of Problem $P12$ satisfies:

(i) $|u_n(\rho,t)| \le CA\rho^{1/2}|\ln\rho|$ *if n is an even number;*

(ii) $|u_n(\rho,t)| \le CA$ *if n is an odd number.*

In both cases the constant C does not depend on the function $f_n(\rho,t)$.

Proof. Let us define the function $g(\xi,\eta) = \rho^{1/2}f_n(\rho,t)$, where $\xi = (\rho+t)/2$, $\eta = (\rho-t)/2$. Then the estimates for f_n imply that g satisfies $|D^{\alpha}g(\xi,\eta)| \le CA\xi^{-|\alpha|}$ for $|\alpha| \le (n-4)_+$. The orthogonality conditions, due to Lemma 2.2, yield

$$\int_0^{\frac{1}{2}}\int_0^{\xi} h^{\nu}_{\nu-2i-2}(\xi,\eta)g(\xi,\eta)\,d\eta d\xi = C\int_0^{\frac{1}{2}}\int_t^{1-t} H^n_i(\rho,t)f_n(\rho,t)\rho d\rho dt = 0.$$

Now Theorem 5.1 gives the required estimates for the solution $v(\xi,\eta)$ of Problem $P13$, given by (3.9), and $u = \rho^{-1/2}v$ is the solution of Problem $P12$. □

Using the similar arguments, we get the corresponding result for the case when not all of the orthogonality conditions (5.10) are fulfilled:

Theorem 5.4. *Let $n,q \in \mathbb{N}\cup\{0\}$, $n \ge 2$, $q \le \left[\frac{n}{2}\right]-1$. Suppose that the function $f_n \in C^{(2q-2)_+}(\overline{\Omega}_0\backslash O)$, $|D^{\alpha}f_n(\rho,t)| \le A\rho^{-|\alpha|-1/2}$ for $|\alpha| \le (2q-2)_+$ and*

$$\int_0^{\frac{1}{2}}\int_t^{1-t} H^n_i(\rho,t)f_n(\rho,t)\rho d\rho dt = 0 \quad \textit{for } i = 0,1,...,q-1. \tag{5.11}$$

Then for the solution $u_n(\rho, t)$ of Problem P12 the upper estimate

$$|u_n(\rho, t)| \leq CA\rho^{-(n-2q-1)} \tag{5.12}$$

holds, where the constant C is independent of $f_n(\rho, t)$. If we suppose also that

$$\int_0^{\frac{1}{2}} \int_t^{1-t} H_q^n(\rho, t) f_n(\rho, t) \rho d\rho dt \neq 0, \tag{5.13}$$

then the lower estimate

$$|u_n(\rho, \rho)| \geq c\rho^{-(n-2q-1)}$$

holds for $c > 0$ and sufficiently small ρ.

Proof. We again define $g(\xi, \eta) = \rho^{1/2} f_n(\rho, t)$ and we prove the theorem applying Theorem 5.2 for $v = \rho^{1/2} u$ and g instead of Theorem 5.1 as in the proof of Theorem 5.3. □

Finally, Theorems 5.3 and 5.4 show that every solution of Problem $P12$ is a linear combination of at most $\left[\frac{n}{2}\right]$ fixed singular solutions:

Lemma 5.3. *For $n \geq 2$ there exist $\left[\frac{n}{2}\right]$ functions $v_n^i(\rho, t)$, $i = 0, ..., \left[\frac{n}{2}\right] - 1$, such that for every generalized solution $u_n(\rho, t)$ of Problem P12 with some right-hand side function $f_n \in C^{(n-4)+}(\overline{\Omega}_0)$ the equality*

$$u_n(\rho, t) = \sum_{i=0}^{\left[\frac{n}{2}\right]-1} c_i v_n^i(\rho, t) + w(\rho, t)$$

holds with some constants c_i and some bounded function $w(\rho, t)$ dependent on $u_n(\rho, t)$.

Proof. Let $u_n(\rho, t)$ be the generalized solution of Problem $P12$ with some right-hand side function $f_n(\rho, t) \in C^{(n-4)+}(\overline{\Omega}_0)$. In general, u_n has a singularity at the origin O. Let k be an integer, $0 \leq k \leq \left[\frac{n}{2}\right] - 1$, and $f_n^{(k)}(\rho, t)$ be the projection of f_n on the linear space $L_{k,n}$ of functions, orthogonal to the functions H_i^n for $i = 0, 1, ..., k$:

$$L_{k,n} := \left\{ f(\rho, t) \;:\; \int_0^{\frac{1}{2}} \int_t^{1-t} H_i^n(\rho, t) f(\rho, t) \rho d\rho dt = 0 \;\text{ for } i = 0, 1, ..., k \right\}.$$

Then $f_n - \sum\limits_{i=0}^{k} \alpha_i^k H_i^n = f_n^{(k)} \in L_{k,n}$ with some constants α_i^k such that

$$\int_0^{\frac{1}{2}} \int_t^{1-t} H_j^n(\rho, t) f_n^{(k)}(\rho, t) \rho d\rho dt = 0, \quad j = 0, ..., k,$$

i.e.

$$\sum_{i=0}^{k} \alpha_i^k \int_0^{\frac{1}{2}} \int_t^{1-t} H_j^n(\rho,t) H_i^n(\rho,t) \rho d\rho dt = \int_0^{\frac{1}{2}} \int_t^{1-t} H_j^n(\rho,t) f_n(\rho,t) \rho d\rho dt$$

for $j = 0, 1, ..., k$. This system has an unique solution for constants α_i^k . To show this, suppose that the rank of the $(k \times k)$-matrix with elements $\int_0^{\frac{1}{2}} \int_t^{1-t} H_j^n H_i^n \rho d\rho dt$ is less than k. Then there are numbers β_0, β_1,...,β_k such that at least one is not zero and

$$\sum_{i=0}^{k} \beta_i \int_0^{\frac{1}{2}} \int_t^{1-t} H_j^n(\rho,t) H_i^n(\rho,t) \rho d\rho dt = 0 \quad \text{for } j = 0, 1, ..., k$$

or

$$\int_0^{\frac{1}{2}} \int_t^{1-t} H_j^n(\rho,t) \left(\sum_{i=0}^{k} \beta_i H_i^n(\rho,t) \right) \rho d\rho dt = 0 \quad \text{for } j = 0, 1, ..., k.$$

Therefore $\sum_{i=0}^{k} \beta_i H_i^n = 0$ in Ω_0, which is impossible, because the functions H_i^n are linearly independent in view of Remark 2.3.

Let us denote by $v_n^i(\rho,t)$ the solution of the Problem $P12$ with a right-hand side function $H_i^n(\rho,t)$, and by $u_n^{(k)}(\rho,t)$ the solution with a right-hand side function $f_n^{(k)}(\rho,t)$. Then we have for u_n the representation

$$u_n = \sum_{i=0}^{k} \alpha_i^k v_n^i + u_n^{(k)}.$$

When $k < \left[\frac{n}{2}\right] - 1$, Theorem 5.4 gives the estimate

$$|u_n^{(k)}(\rho,t)| \le C\rho^{-(n-2k-3)},$$

while for $k = k_0 := \left[\frac{n}{2}\right] - 1$ Theorem 5.3 shows that the function $w := u_n^{(k_0)}$ is at least bounded. $\square$

6. PROOF OF THE MAIN RESULTS

We are now ready to prove Theorem 1.1 and Theorem 1.2 (see Introduction).

Proof of Theorem 1.1. When the right-hand side function has the form

$$f = f_0^1(\rho,t) + \sum_{n=1}^{l} \left(f_n^1(\rho,t) \cos n\varphi + f_n^2(\rho,t) \sin n\varphi \right),$$

one could construct the unique generalized solution $u(x_1, x_2, t)$ as

$$u = u_0^1(\rho, t) + \sum_{n=1}^{l} \left(u_n^1(\rho, t) \cos n\varphi + u_n^2(\rho, t) \sin n\varphi \right), \tag{6.1}$$

where the functions u_n^i are solutions of Problem $P12$ with right-hand side function $f_n^i \in C^{(l-4)+}(\overline{\Omega}_0)$. First of all, we use Theorem 4.1 to estimate the functions u_0^1, u_1^1 and u_1^2:

$$|u_0^1(\rho, t)| \le C\rho^{1/4} \max_{\overline{\Omega}_0} |\rho^{1/2} f_0^1| \le Cr^{1/4} ||f_0||_0^*,$$

$$|u_1^i(\rho, t)| \le C_\delta \rho^{-\delta} \max_{\overline{\Omega}_0} |\rho^{1/2} f_1^i| \le C'_\delta r^{-\delta} ||f_1||_0^*.$$

For the case $n \ge 2$ we apply Theorem 5.3 with the constant $A = ||f_n||_{(n-4)_+}^*$. Because of the identity

$$\int_0^{\frac{1}{2}} \int_t^{1-t} H_k^n(\rho, t) f_n^i(\rho, t) \rho d\rho dt = c \int_\Omega V_k^{n,i}(x_1, x_2, t) f(x_1, x_2, t) dx_1 dx_2 dt = 0, \tag{6.2}$$

the orthogonality conditions (5.10) are fulfilled. Therefore, if n is an even number,

$$|u_n^i| \le C_n A \rho^{1/2} |\ln \rho| \le C'_n r^{1/2} |\ln r| ||f_n||_{(n-4)_+}^*,$$

while

$$|u_n^i| \le C_n A = C_n ||f_n||_{(n-4)_+}^*$$

if n is an odd number. Finally, summing up all these inequalities, we find

$$\begin{aligned} |u| \le \quad & |u_0^1| + \sum_{n=1}^{l} \left(|u_n^1| + |u_n^2| \right) \le C_1 r^{1/4} ||f_0||_0^* + C_{2,\delta} r^{-\delta} ||f_1||_0^* \\ & + C_3 r^{1/2} |\ln r| \sum_{k=1}^{[l/2]} ||f_{2k}||_{(2k-4)_+}^* + C_4 \sum_{k=1}^{\left[\frac{l-1}{2}\right]} ||f_{2k+1}||_{(2k-3)_+}^*. \qquad \square \end{aligned}$$

Remark 6.1. Notice that the orthogonality condition $\int_\Omega V_k^{n,i} f dx_1 dx_2 dt = 0$ for a function f with the representation (1.3) is imposed only on the function $f_n^i(\rho, t)$ and has no influence over the other functions $f_m^j(\rho, t)$ from (1.3) with indices $(m, j) \ne (n, i)$.

Proof of Theorem 1.2. We use again the representation (6.1). For $n = 0$ and 1 we have respectively

$$|u_0^1| \le C\rho^{1/4} ||f_0||_0^* \le C' r^{-s+1} ||f_0||_0^*$$

and

$$|u_1^i| \le C_{1/2} \rho^{-1/2} ||f_1||_0^* \le C'' r^{-s+1} ||f_1||_0^*,$$

because $s \ge 2$. For $n \ge 2$ we apply Theorem 5.4 with $q = \left[\dfrac{n - s + 1}{2}\right]_+$ and $A = ||f_n||_{2q-2}^*$. Now, the identity (6.2) shows that the orthogonality conditions hold for $n - 2k \geq s + 1$, i.e. for all $k \geq 0$ such that $q - 1 \geq \left[\frac{n-s+1}{2}\right] - 1 \geq \left[\frac{2k+2}{2}\right] - 1 = k$. Therefore

$$|u_n^i| \leq CA\rho^{-n+2q+1} \leq C'\rho^{-s+1}||f_n||^*_{(2q-2)_+} \leq C''r^{-s+1}||f_n||^*_{(n-s-1)_+},$$

because depending of parity of n we have $-n + 2\left[\frac{n-s+1}{2}\right] + 1 = -s + 1$ or $-s + 2$, and $2\left[\frac{n-s+1}{2}\right] - 2 = n - s - 1$ or $n - s - 2$. These give the required upper estimate for the solution u. For the second part of the theorem, let us notice that when $m - 2p = s$, the corresponding number q is $q = \left[\frac{m-s+1}{2}\right] = p$. Thus, the lower estimate in Theorem 5.4 gives

$$|u_m^j|_{t=\rho}| \geq c\rho^{-m+2q+1} = c\rho^{-s+1},$$

which completes the proof. □

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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THE TRIGONOMETRIC ANALOGUE OF THE TAYLOR'S FORMULA AND ITS APPLICATION

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A new approach to establishing generalized Taylor's expansions is used to prove the trigonometric analogue of the Taylor's formula. We derive point-wise estimates of the error in the trigonometric interpolation and approximation by convolutional linear operators.

Keywords: Taylor's formula, trigonometric interpolation, convolutional operators
2000 MSC : 42A15, 42A85

1. INTRODUCTION

We consider the function spaces $L_p^*[-\pi,\pi]$, $1 \le p < \infty$, and $C^*[-\pi,\pi]$, where

$$L_p^*[-\pi,\pi] = \{f : \mathbf{R} \to \mathbf{R} : f(x+2\pi) = f(x) \text{ a.e., } f|_{[-\pi,\pi]} \in L_p[-\pi,\pi]\},$$
$$C^*[-\pi,\pi] = \{f \in C(\mathbf{R}) : f(x+2\pi) = f(x)\},$$

normed, respectively, with the usual L_p-norm over the interval $[-\pi,\pi]$ for $1 \le p < \infty$, denoted by $\|\cdot\|_p$, and the uniform norm over the interval $[-\pi,\pi]$, denoted by $\|\cdot\|_\infty$.

In a recent paper (see [1]) we have introduced a new modulus of smoothness, which describes the rate of the best trigonometric approximation. It is defined by

$$\omega_r^T(f;t)_p := \sup_{0<h\le t} \|\Delta_h^{2r-1}\mathcal{F}_{r-1}f\|_p,\ r = 1,2,\ldots,$$

where

$$\Delta_h^{2r-1} f(x) := \sum_{k=0}^{2r-1} (-1)^k \binom{2r-1}{k} f(x + ((2r-1)/2 - k)h)$$

is the symmetric finite difference of order $2r - 1$,

$$\mathcal{F}_{r-1}(f, x) = f(x) + \int_0^x \mathcal{K}_{r-1}(t) f(x-t)\, dt$$

and

$$\mathcal{K}_{r-1}(t) = \sum_{j=1}^{r-1} \frac{a_j^{(r-1)}}{(2j-1)!} t^{2j-1}, \quad a_j^{(r-1)} = \sum_{1 \le l_1 < \cdots < l_j \le r-1} (l_1 \cdots l_j)^2.$$

It is shown in [1] that for the rate of the best trigonometric approximation $E_n^T(f)_p := \inf_{\tau \in T_n} \|f - \tau\|_p$, T_n being the set of all trigonometric polynomials of degree at most n, we have

$$E_n^T(f)_p \le C_r \omega_r^T(f; n^{-1})_p, \quad n \ge r-1, \tag{1.1}$$

and

$$\omega_r^T(f; t)_p \le C_r t^{2r-1} \sum_{r-1 \le k \le 1/t} (k+1)^{2r-2} E_k^T(f)_p, \quad 0 < t \le \frac{1}{r}. \tag{1.2}$$

Moreover, we have $\omega_r^T(f; t)_p \equiv 0$ if and only if $f \in T_{r-1}$. In that sense the new modulus of smoothness describes the rate of the best trigonometric approximation more precisely than the classical one. The modulus of smoothness $\omega_r^T(f; t)_p$ possesses properties similar to those of the classical one, as it is shown in [1].

Let $L_n : L_p^*[-\pi, \pi] \to L_p^*[-\pi, \pi]$, $1 \le p < \infty$, or $L_n : C^*[-\pi, \pi] \to C^*[-\pi, \pi]$, be a bounded linear operator that preserves the trigonometric polynomials of degree n. Then the well-known Lebesgue inequality

$$\|f - L_n f\|_p \le (1 + \|L_n\|) E_n^T(f)_p$$

and the Jackson-type estimate (1.1) imply

$$\|f - L_n f\|_p \le C_r (1 + \|L_n\|)\, \omega_r^T(f, n^{-1})_p, \quad n \ge r-1.$$

Similar estimates, using the classical periodic modulus of smoothness, are known. For instance, G. P. Nevai has proved in [3] the following generalization of a result of S. M. Nikolskii:

$$\|f - t_n f\|_\infty \le 2^{-r} \omega_r\Big(f; \frac{2\pi}{2n+1}\Big)_\infty \lambda_n(\bar{x}) + \mathcal{O}\big(\omega_r(f; n^{-1})_\infty\big),$$

where $t_n f \in T_n$ interpolates the 2π-periodic continuous function f in the equidistant nodes $\bar{x} = (x_{-n}, \ldots, x_n)$, $x_k = 2k\pi/(2n+1)$, $k = -n, \ldots, n$, and $\lambda_n(\bar{x})$ is the Lebesgue constant for the trigonometric Lagrange interpolation. For similar estimates in uniform norm, concerning the approximation by the partial sums of the Fourier series, one can refer to [2] and [4].

The trigonometric analogue of the Taylor's formula will allow us to derive a point-wise estimate of the error $f(x) - L_n(f, x)$ for a smooth f. We need to introduce several notations to state that result. We define the differential operators

$$D_j = \left(\frac{d}{dx}\right)^2 + j^2 I, \quad j = 1, 2, \ldots, \tag{1.3}$$

where I is the identity. We also put

$$\begin{aligned} \widetilde{D}_{n+1} &= D_n \cdots D_1 \frac{d}{dx}, \\ \widehat{D}_{n0} &= D_1 \cdots D_n, \\ \widehat{D}_{nk} &= D_1 \cdots D_{k-1} D_{k+1} \cdots D_n, \quad k = 1, \ldots, n. \end{aligned}$$

Let us observe that $\widetilde{D}_{n+1} g = 0$, $g \in C^{2n+1}[a, b]$, if and only if $g \in T_n$ in $[a, b]$. The following trigonometric analogue of the Taylor's formula holds true (see [5, §10.8]).

Theorem 1 (Taylor's trigonometric formula). *Let $f \in C^{2n+1}(\Delta_c)$, where Δ_c is any of the intervals $[c, c+\delta]$, $[c-\delta, c]$ or $[c-\delta, c+\delta]$ for $c \in \mathbf{R}$ and $\delta > 0$, and let also*

$$\begin{aligned} \tau_{n,c}(f, x) = {} & \frac{\widehat{D}_{n0} f(c)}{(n!)^2} + 2 \sum_{k=1}^{n} \frac{(-1)^{k-1}}{(n-k)!\,(n+k)!} \\ & \times \left[(k^2 \widehat{D}_{nk} f(c) - \widehat{D}_{n0} f(c)) \cos k(x-c) + k \widehat{D}_{nk} f'(c) \sin k(x-c) \right]. \end{aligned} \tag{1.4}$$

Then $\tau_{n,c} f \in T_n$, $\tau_{n,c}^{(s)}(f, c) = f^{(s)}(c)$, $s = 0, 1, \ldots, 2n$, and for $x \in \Delta_c$ we have

$$f(x) = \tau_{n,c}(f, x) + \frac{1}{n!\,(2n-1)!!} \int_c^x (1 - \cos(x-t))^n \widetilde{D}_{n+1} f(t)\, dt. \tag{1.5}$$

Let $-\pi \le x_0 < \cdots < x_{2n} < \pi$ be arbitrary nodes. Let us denote by $t_n(f, x)$ the unique trigonometric polynomial of degree n, which interpolates f in those nodes. Then the theorem above easily implies a point-wise estimate of the error $f(x) - t_n(f, x)$ for a smooth function f.

Proposition 2. *Let $f \in C^{2n+1}[-\pi, \pi]$. Then*

$$f(x) - t_n(f, x) = \frac{1}{n!\,(2n-1)!!} \int_{-\pi}^{\pi} K(x, t) \widetilde{D}_{n+1} f(t)\, dt, \quad x \in [-\pi, \pi],$$

where

$$K(x, t) = \left(1 - \cos[(x-t)_+]\right)^n - \sum_{k=0}^{2n} \left(1 - \cos[(x_k - t)_+]\right)^n t_{nk}(x)$$

and $(x-t)_+ = \max\{x - t, 0\}$.

The contents of the paper are organized as follows. In Section 2 we collect few auxiliary results, which are necessary for the proof of the Taylor's trigonometric formula, presented in Section 3. Finally, in the last section we derive point-wise estimates of the error in the trigonometric interpolation and in the approximation by convolutional linear operators.

2. AUXILIARY RESULTS

Let $[a, b]$ be a finite interval such that $0 \in [a, b]$. We define the convolutional operator, known as Duhamel's convolution, $\circledast : L_1[a, b] \times L_1[a, b] \to L_1[a, b]$,

$$f \circledast g(x) := \int_0^x f(x - t)g(t)\, dt.$$

It is easy to verify that it possesses the properties:

1. $f \circledast g = g \circledast f$;
2. $f \circledast (g + h) = f \circledast g + f \circledast h$;
3. $f \circledast (g \circledast h) = (f \circledast g) \circledast h$.

Next we introduce a number of notations. We put $\varphi_n(x) = \sin nx,\ n = 1, 2, \dots,$ and $\Phi_n = \varphi_1 \circledast \cdots \circledast \varphi_n$, $\widetilde{\Phi}_n = \Phi_n \circledast 1$, $\widehat{\Phi}_n = \widetilde{\Phi}_n \circledast 1$. The propositions bellow contain some of the properties of Φ_n, $\widetilde{\Phi}_n$ and $\widehat{\Phi}_n$, but first we prove the following simple lemma.

Lemma 3. *Any function of the form*

$$f(x) = cx + a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) \tag{2.1}$$

has at most $2n + 1$ zeroes in $[-\pi, \pi)$, counting the multiplicities, that is, $x, 1, \cos x, \sin x, \dots, \cos nx, \sin nx$ is an extended Chebyshev system in $[-\pi, \pi)$. Hence, for any choice of $-\pi \le x_1 < \cdots < x_m < \pi$ and positive integers $\nu_1, \dots, \nu_m$ with $\nu_1 + \cdots + \nu_m = 2n + 1$ there exists only one function of the form (2.1) with a fixed c for which x_k is a zero of multiplicity ν_k, $k = 1, \dots, m$.

Proof. It is enough to prove the first part of the statement. We follow a standard argument assuming the opposite and making use of the well-known Rolle's theorem. So, let us assume that $f(x)$ has at least $2n+2$ zeroes in $[-\pi, \pi)$, counting the multiplicities. Then $f'(x)$ has at least $2n + 1$ zeroes in $[-\pi, \pi)$, counting the multiplicities. But $f'(x)$ is a trigonometric polynomial of degree n and therefore it has at most $2n$ zeroes in $[-\pi, \pi)$, counting the multiplicities. This contradiction verifies the statement of the lemma. □

Proposition 4. *We have*
(i) $D_n\Phi_n = n\Phi_{n-1}$ *and* $D_n\widehat{\Phi}_n = n\widehat{\Phi}_{n-1}$ *for* $n = 2, 3, \dots$;
(ii) $\Phi_n(x) = c_n \sin x(1-\cos x)^{n-1}$, *where* $c_n = \dfrac{n}{(2n-1)!!}$, $n = 1, 2\dots$;
(iii) $\widetilde{\Phi}_n(x) = \dfrac{1}{(2n-1)!!}(1-\cos x)^n$;
(iv) $\widehat{\Phi}_n(x) = \dfrac{x}{n!} + \sum_{k=1}^{n} b_{nk}\sin kx$,
where $\{b_{nk}\}$ *is the unique solution of the linear system*

$$\left|\begin{array}{l} \sum_{k=1}^{n} k b_{nk} = -\dfrac{1}{n!}, \\ \sum_{k=1}^{n} k^s b_{nk} = 0, \ s = 3, 5, \dots, 2n-1. \end{array}\right.$$

Proof. The first statement of the proposition follows by differentiation of the recursion relation $\Phi_n = \varphi_n \circledast \Phi_{n-1}$. Namely, we have

$$\begin{aligned}\left(\frac{d}{dx}\right)^2 \Phi_n(x) &= \left(\frac{d}{dx}\right)^2 \int_0^x \sin n(x-t)\Phi_{n-1}(t)\,dt = n\frac{d}{dx}\int_0^x \cos n(x-t)\Phi_{n-1}(t)\,dt \\ &= n\Phi_{n-1}(x) - n^2\int_0^x \sin n(x-t)\Phi_{n-1}(t)\,dt \\ &= n\Phi_{n-1}(x) - n^2\Phi_n(x).\end{aligned}$$

Thus we have got $D_n\Phi_n = n\Phi_{n-1}$. If we put $e_1(x) = x$, then $\widehat{\Phi}_n = \widetilde{\Phi}_n \circledast 1 = \Phi_n \circledast 1 \circledast 1 = \Phi_n \circledast e_1$. Therefore $\widehat{\Phi}_n$ satisfies the same recursion relation as Φ_n with $\widehat{\Phi}_1(x) = x - \sin x$ instead of $\Phi_1(x) = \sin x$. Hence we get $D_n\widehat{\Phi}_n = n\widehat{\Phi}_{n-1}$. This completes the proof of (i).

To verify (ii), we consider the sequence of trigonometric polynomials

$$P_n(x) = c_n \sin x(1-\cos x)^{n-1}, \ c_n = \frac{n}{(2n-1)!!}, \ n \geq 1.$$

We shall show that it satisfies the same recursion relation as Φ_n in (i) and $P_n(0) = 0 = \Phi_n(0)$, $P_n'(0) = 0 = \Phi_n'(0)$, $n \geq 2$. Hence, as $P_1 = \Phi_1$, we have $P_n = \Phi_n$, $n \geq 2$. For $n \geq 2$

$$\begin{aligned}P_n''(x) &= c_n\left(\sin x(1-\cos x)^{n-1}\right)'' \\ &= c_n \sin x(1-\cos x)^{n-2}(n^2 - 3n + 1 + n^2\cos x).\end{aligned}$$

Consequently,

$$\begin{aligned}
D_n P_n(x) &= P_n''(x) + n^2 P_n(x) \\
&= c_n \sin x(1-\cos x)^{n-2}(n^2 - 3n + 1 + n^2 \cos x) \\
&\qquad\qquad\qquad\qquad + n^2 c_n \sin x(1-\cos x)^{n-1} \\
&= c_n(2n^2 - 3n + 1)\sin x(1-\cos x)^{n-2} \\
&= nc_{n-1}\sin x(1-\cos x)^{n-2} \\
&= nP_{n-1}(x).
\end{aligned}$$

We get (iii) by integrating (ii).

It remains to verify (iv). From (i) it follows $\widetilde{D}_{n+1}\widehat{\Phi}_n = n!$. Consequently, $\widehat{\Phi}_n(x) = x/n! + a_0 + \sum_{k=1}^{n}(a_{nk}\cos kx + b_{nk}\sin kx)$ for some constants a_{nk}, b_{nk}. Assertion (iii) implies that $\widetilde{\Phi}_n$ is an even function, therefore $\widehat{\Phi}_n$ is an odd one. This implies that $\widehat{\Phi}_n(x) = x/n! + \sum_{k=1}^{n} b_{nk}\sin kx$ for some $b_{nk} \in \mathbf{R}$. Next we have $\widehat{\Phi}_n'(0) = \widetilde{\Phi}_n(0) = 0$, which implies

$$\sum_{k=1}^{n} kb_{nk} = -\frac{1}{n!}.$$

It is easy to see that $\widehat{\Phi}_n^{(s)}(0) = 0$, $s = 2, \ldots, 2n-1$, as well. For s even this is obvious. For s odd we can verify it, for instance, by induction on n. For $n = 1$ the statement is trivial as we have shown above. We assume that $\widehat{\Phi}_n^{(s)}(0) = 0$, $s = 1, \ldots, 2n-1$, and shall verify it for $n+1$ in the place of n. We differentiate in x *the equality* $D_{n+1}\widehat{\Phi}_{n+1}(x) = (n+1)\widehat{\Phi}_n(x)$ *and get for* $s = 1, \ldots, 2n-1$

$$\widehat{\Phi}_{n+1}^{(s+2)}(x) + (n+1)^2\widehat{\Phi}_{n+1}^{(s)}(x) = (n+1)\widehat{\Phi}_n^{(s)}(x).$$

Then, putting $x = 0$, we get $\widehat{\Phi}_{n+1}^{(s)}(0) = 0$ consecutively for $s = 3, 5, \ldots, 2n+1$, which is what we had to show. Now

$$\sum_{k=1}^{n} k^s b_{nk} = 0, \ s = 3, 5, \ldots, 2n-1,$$

follows from $\widehat{\Phi}_n^{(s)}(0) = 0$, $s = 3, \ldots, 2n-1$ $(n > 1)$. In passing, let us note that the linear system

$$\left|\begin{aligned}
&\sum_{k=1}^{n} kb_{nk} = -\frac{1}{n!}, \\
&\sum_{k=1}^{n} k^s b_{nk} = 0, \ s = 3, 5, \ldots, 2n-1
\end{aligned}\right.$$

has a unique solution due to Lemma 3. This completes the proof of (iv). □

The following representation of $\widehat{\Phi}_n(x)$ has been pointed out to the author by K. G. Ivanov.

Proposition 5. *The following formula holds :*

$$\widehat{\Phi}_n(x) = \frac{1}{n!}\Big(x - \sum_{k=1}^{n} \frac{(k-1)!}{(2k-1)!!} \sin x (1-\cos x)^{k-1}\Big). \tag{2.2}$$

Proof. We just write for $n \geq 1$

$$\begin{aligned} J_n(x) := \int_0^x \sin^{2n} t\, dt &= -\int_0^x \sin^{2n-1} t\, d\cos t \\ &= -\sin^{2n-1} x \cos x + (2n-1)\int_0^x \cos^2 t \sin^{2(n-1)} t\, dt \\ &= -\sin^{2n-1} x \cos x + (2n-1)\int_0^x \sin^{2(n-1)} t\, dt - (2n-1)\int_0^x \sin^{2n} t\, dt. \end{aligned}$$

Therefore

$$J_n(x) = -\sin^{2n-1} x \cos x + (2n-1)J_{n-1}(x) - (2n-1)J_n(x).$$

Hence we get the recursion relation

$$J_n(x) = -\frac{1}{2n}\sin^{2n-1} x \cos x + \frac{2n-1}{2n} J_{n-1}(x), \quad n \geq 1.$$

Consequently, noting that $J_0(x) = x$, we get

$$\begin{aligned} J_n(x) &= \frac{1}{2n}\Big(\frac{(2n-1)!!}{(2n-2)!!}x - \sin^{2n-1} x \cos x \\ &\qquad - \sum_{l=1}^{n-1} \frac{(2n-1)(2n-3)\cdots(2n-2l+1)}{(2n-2)(2n-4)\cdots(2n-2l)} \sin^{2n-2l-1} x \cos x\Big) \\ &= \frac{1}{2n}\Big(\frac{(2n-1)!!}{(2n-2)!!}x - \frac{1}{2}\sin^{2(n-1)} x \sin 2x \\ &\qquad - \frac{(2n-1)!!}{2(2n-2)!!}\sum_{l=1}^{n-1} \frac{(2n-2l-2)!!}{(2n-2l-1)!!} \sin^{2(n-l-1)} x \sin 2x\Big) \\ &= \frac{(2n-1)!!}{(2n)!!}\Big(x - \frac{(2n-2)!!}{2(2n-1)!!}\sin^{2(n-1)} x \sin 2x \\ &\qquad - \frac{1}{2}\sum_{l=1}^{n-1} \frac{(2n-2l-2)!!}{(2n-2l-1)!!} \sin^{2(n-l-1)} x \sin 2x\Big) \\ &= \frac{(2n-1)!!}{(2n)!!}\Big(x - \frac{1}{2}\sum_{l=0}^{n-1} \frac{(2n-2l-2)!!}{(2n-2l-1)!!} \sin^{2(n-l-1)} x \sin 2x\Big) \\ &= \frac{(2n-1)!!}{(2n)!!}\Big(x - \frac{1}{2}\sum_{k=1}^{n} \frac{(2k-2)!!}{(2k-1)!!} \sin^{2(k-1)} x \sin 2x\Big) \end{aligned}$$

$$= \frac{(2n-1)!!}{(2n)!!}\Big(x - \frac{1}{2}\sum_{k=1}^{n} \frac{(k-1)!}{(2k-1)!!} 2^{k-1} \sin^{2(k-1)} x \sin 2x\Big)$$
$$= \frac{(2n-1)!!}{2^{n+1} n!}\Big(2x - \sum_{k=1}^{n} \frac{(k-1)!}{(2k-1)!!}(1-\cos 2x)^{k-1} \sin 2x\Big).$$

Thus we have shown

$$J_n(x) = \frac{(2n-1)!!}{2^{n+1} n!}\Big(2x - \sum_{k=1}^{n} \frac{(k-1)!}{(2k-1)!!} \sin 2x (1-\cos 2x)^{k-1}\Big). \tag{2.3}$$

To finish the proof, we just write

$$\widehat{\Phi}_n(x) = \frac{1}{(2n-1)!!}\int_0^x (1-\cos t)^n \, dt = \frac{2^{n+1}}{(2n-1)!!}\int_0^{x/2} \sin^{2n} t \, dt$$
$$= \frac{2^{n+1}}{(2n-1)!!} J_n(x/2).$$

Hence, making use of (2.3), we get (2.2). □

Let $[a,b]$ be a finite interval such that $0 \in [a,b]$. In [1] we have proved that $\mathcal{F}_n : C[a,b] \to C[a,b]$ can be represented in the form

$$\mathcal{F}_n = A_1 \cdots A_n,$$

where the bounded linear operators $A_j : C[a,b] \to C[a,b]$, $j = 1, 2, \ldots$, are defined by

$$A_j(f,x) := f(x) + j^2 \int_0^x (x-t) f(t) \, dt, \quad j = 1, 2, \ldots$$

In the above mentioned investigation we have also shown the following assertion.

Proposition 6. *The bounded linear operator A_j is invertible and*

$$A_j^{-1}(g,x) = g(x) - j \int_0^x \sin j(x-t) g(t) \, dt.$$

Hence

$$A_j^{-1}(g,x) = \frac{1}{j}\int_0^x \sin j(x-t) g''(t) \, dt$$

for $g \in C^2[a,b]$ with $g(0) = g'(0) = 0$.

Now we are ready to prove formula (1.5).

Proof of Theorem 1 It is enough to prove the assertion of the theorem for $c = 0$. Hence it will follow for any $c \in \mathbf{R}$ by translation. Let $\tau(x) = a_0 + a_1 \cos x + b_1 \sin x + \cdots + a_n \cos nx + b_n \sin nx$ be the unique trigonometric polynomial of degree at most n, which interpolates f in $x = 0$ with multiplicity $2n + 1$, i.e., $\tau^{(s)}(0) = f^{(s)}(0)$ for $s = 0, 1, \dots, 2n$. Using

$$D_j \cos kx = (j^2 - k^2) \cos kx \quad \text{and} \quad D_j \sin kx = (j^2 - k^2) \sin kx,$$

we get

$$\begin{aligned}
&(n!)^2 a_0 = \widehat{D}_{n0} \tau(0) = \widehat{D}_{n0} f(0), \\
&(-1)^{k-1} \frac{(n-k)!\,(n+k)!}{2k^2} a_k + \frac{(n!)^2}{k^2} a_0 = \widehat{D}_{nk} \tau(0) = \widehat{D}_{nk} f(0), \ k = 1, \dots, n, \\
&(-1)^{k-1} \frac{(n-k)!\,(n+k)!}{2k} b_k = \widehat{D}_{nk} \tau'(0) = \widehat{D}_{nk} f'(0), \ k = 1, \dots, n.
\end{aligned}$$

Hence $\tau_{n,0}(f, x) = \tau(x)$.

It remains to consider the remainder $r_n(x) = f(x) - \tau_{n,0}(f, x)$. Let us put for the sake of brevity

$$F(x) = \int_0^x \left(\int_0^{t_1} \cdots \left(\int_0^{t_{2n}} \widetilde{D}_{n+1} f(t_{2n+1})\, dt_{2n+1} \right) \cdots dt_2 \right) dt_1.$$

Obviously, $F \in C^{2n+1}(\Delta_0)$ and $F^{(s)}(0) = 0, \ s = 0, 1, \dots, 2n$. Now $r_n(x) = f(x) - \tau_{n,0}(f, x)$ implies $\widetilde{D}_{n+1} r_n(x) = \widetilde{D}_{n+1} f(x)$, $x \in \Delta_0$. We have proved in [1] that

$$(\mathcal{F}_n g)^{(2n+1)} = \widetilde{D}_{n+1} g, \quad g \in C^{2n+1}(\Delta_0).$$

Therefore $(d/dx)^{2n+1} \mathcal{F}_n(r_n, x) = \widetilde{D}_{n+1} r_n(x)$, $x \in \Delta_0$. Hence, making use of $r_n^{(s)}(0) = 0, \ s = 0, 1, \dots, 2n$, we get $\mathcal{F}_n(r_n, x) = F(x)$, $x \in \Delta_0$, that is,

$$A_1 \cdots A_n r_n = F. \tag{3.1}$$

Proposition 6 states for $g \in C^2(\Delta_0)$ with $g(0) = g'(0) = 0$ that

$$A_j^{-1} g = \frac{1}{j} \varphi_j \circledast g''. \tag{3.2}$$

Simple calculations yield for $g \in C^2(\Delta_0)$ with $g(0) = g'(0) = 0$

$$(\Phi_k \circledast g)'' = \Phi_k \circledast g'', \tag{3.3}$$

and for any $g \in C(\Delta_0)$

$$\Phi_k \circledast g(0) = (\Phi_k \circledast g)'(0) = 0. \tag{3.4}$$

Now (3.1) and (3.2) for $j = 1$ imply

$$A_2 \cdots A_n r_n = \varphi_1 \circledast F'' = \Phi_1 \circledast F''.$$

Next, applying again (3.2) (for $j = 2$), using (3.4) (for $k = 1$), and then (3.3) (for $k = 1$), we have

$$A_3 \cdots A_n r_n = \frac{1}{2}\varphi_1 \circledast \varphi_2 \circledast F^{(4)} = \frac{1}{2}\Phi_2 \circledast F^{(4)}.$$

Proceeding in this way, we finally get

$$r_n = \frac{1}{n!}\Phi_n \circledast F^{(2n)}. \tag{3.5}$$

To finish the proof, we write

$$\begin{aligned} r_n(x) &= \frac{1}{n!}\int_0^x \left(\Phi_n(x-t)\int_0^t \widetilde{D}_{n+1} f(s)\,ds\right) dt \\ &= -\frac{1}{n!}\int_0^x \left(\int_0^t \widetilde{D}_{n+1} f(s)\,ds\right) d\widetilde{\Phi}_n(x-t) \\ &= \frac{1}{n!}\int_0^x \widetilde{\Phi}_n(x-t)\widetilde{D}_{n+1} f(t)\,dt \\ &= \frac{1}{n!}\widetilde{\Phi}_n \circledast \widetilde{D}_{n+1} f(x). \end{aligned}$$

This completes the proof of the theorem as Proposition 4 (iii) states $\widetilde{\Phi}_n(x) = 1/(2n-1)!!\,(1-\cos x)^n$. □

Remark 7. An estimate of the remainder. (Again we discuss the case $c = 0$.) The mean value theorem implies

$$r_n(x) = \frac{\widetilde{D}_{n+1} f(\xi_x)}{n!\,(2n-1)!!}\int_0^x (1-\cos t)^n\,dt, \; x \in \Delta_0, \tag{3.6}$$

where $\xi_x \in \Delta_0$ depends on x. Hence

$$|r_n(x)| \leq \frac{\|\widetilde{D}_{n+1} f\|_{\infty(\Delta_0)}}{n!\,(2n-1)!!}\left|\int_0^x (1-\cos t)^n\,dt\right|, \; x \in \Delta_0. \tag{3.7}$$

Now, using the simple inequality $1 - \cos x \leq x^2/2$, we get

$$|r_n(x)| \leq \frac{|x|^{2n+1}}{(2n+1)!}\|\widetilde{D}_{n+1} f\|_{\infty(\Delta_0)}, \; x \in \Delta_0. \tag{3.8}$$

Formula (1.5) can be useful in expressing the error in approximation by linear operators that preserves trigonometric polynomials up to a given degree. Indeed, let $L_n : C[-\pi,\pi] \to C[-\pi,\pi]$ be such that $L_n f = f$ if $f \in T_n$ and let $f \in C^{2n+1}[-\pi,\pi]$. Then we have

$$f - L_n f = (I - L_n) r_n f, \tag{4.1}$$

where

$$r_n(f,x) = \frac{1}{n!\,(2n-1)!!} \int_c^x (1-\cos(x-t))^n \widetilde{D}_{n+1} f(t)\,dt$$

for some fixed $c \in [-\pi,\pi]$.

Let $-\pi \le x_0 < \cdots < x_{2n} < \pi$ be arbitrary nodes. Then, as it is known, there exists a unique trigonometric polynomial $t_n(f,x)$ of degree n such that $t_n(f,x_k) = f(x_k)$, $k = 0,\ldots,2n$. It can be represented in the form

$$t_n(f,x) = \sum_{k=0}^{2n} f(x_k) t_{nk}(x), \tag{4.2}$$

where

$$t_{nk}(x) = \frac{\prod_{j=0,j\neq k}^{2n} \sin\dfrac{x-x_j}{2}}{\prod_{j=0,j\neq k}^{2n} \sin\dfrac{x_k-x_j}{2}}. \tag{4.3}$$

Now the considerations in the beginning of this section and (1.5) with $c = -\pi$ easily yield Proposition 2. That proposition implies the following estimates of the error $f(x) - t_n(f,x)$ for smooth functions f.

Corollary 8. *Let $f \in C^{2n+1}[-\pi,\pi]$. Then we have for $x \in [-\pi,\pi]$*

(i) $|f(x) - t_n(f,x)| \le \dfrac{\pi\mu(\bar{x})\|\widetilde{D}_{n+1} f\|_\infty}{2^n(n-1)!\,(2n-1)!!} |(x-x_0)\ldots(x-x_{2n})|,$

where

$$\mu(\bar{x}) = \sum_{k=0}^{2n} \left(\prod_{j=0,j\neq k}^{2n} \left|\sin\frac{x_k-x_j}{2}\right| \right)^{-1}.$$

(ii) $|f(x) - t_n(f,x)| \le \dfrac{2^{n+1}\pi^2\mu(\bar{x})\|\widetilde{D}_{n+1} f\|_\infty}{a\,(n-1)!\,(2n-1)!!} \left|\sin\dfrac{x-x_0}{2}\cdots\sin\dfrac{x-x_{2n}}{2}\right|$

for nodes $-\pi + a \le x_0 < \cdots < x_{2n} \le \pi - a$, $a \in (0,\pi)$.

Proof. The assertions of the corollary follow easily from the estimate

$$\begin{aligned}\big|(1-\cos[(x-t)_+])^n - (1-\cos[(x_k-t)_+])^n\big| \\ \le n2^{n-1}|\cos[(x_k-t)_+] - \cos[(x-t)_+]|\end{aligned} \tag{4.4}$$

and the relation

$$\cos[(x_k-t)_+]-\cos[(x-t)_+] = \begin{cases} -2\sin\dfrac{x+x_k-2t}{2}\sin\dfrac{x_k-x}{2}, & t\le x, x_k, \\ 2\sin^2\dfrac{x-t}{2}, & x_k\le t\le x, \\ -2\sin^2\dfrac{x_k-t}{2}, & x\le t\le x_k, \\ 0, & t\ge x, x_k. \end{cases} \tag{4.5}$$

Now (4.4), (4.5) and the inequality $|\sin x|\le|x|$ imply

$$|(1-\cos[(x-t)_+])^n-(\cos[(x_k-t)_+])^n|\le n2^{n-1}|x-x_k|,$$

therefore, using again the inequality $|\sin x|\le|x|$ and the fact that $\sum_{k=0}^{2n} t_{nk}(x)\equiv 1$, we get for any x and t

$$|K(x,t)|\le n2^{n-1}\sum_{k=0}^{2n}|x-x_k||t_{nk}(x)|\le n2^{-n-1}\mu(\bar{x})|x-x_0|\cdots|x-x_{2n}|.$$

Hence assertion (i) follows. To verify (ii), we just have to notice that if $-\pi+a\le x_0<\cdots<x_{2n}\le\pi-a$, where $a\in(0,\pi)$, and $x\in[-\pi,\pi]$, then

$$\sin\frac{x-t}{2}\le\frac{\sin\dfrac{x-x_k}{2}}{\sin\dfrac{a}{2}},\ x_k\le t\le x, \quad\text{and}\quad \sin\frac{x_k-t}{2}\le\frac{\sin\dfrac{x_k-x}{2}}{\sin\dfrac{a}{2}},\ x\le t\le x_k.$$

These two estimates, (4.4), (4.5) and the inequality $|\sin x|\ge(2/\pi)|x|$, $|x|\le\pi/2$, yield for $x\in[-\pi,\pi]$ and any t

$$|(1-\cos[(x-t)_+])^n-(\cos[(x_k-t)_+])^n|\le\frac{n2^n\pi}{a}\left|\sin\frac{x-x_k}{2}\right|,$$

which, on its turn, implies for $x\in[-\pi,\pi]$ and any t

$$\begin{aligned}|K(x,t)|&\le\frac{n2^n\pi}{a}\sum_{k=0}^{2n}\left|\sin\frac{x-x_k}{2}\right||t_{nk}(x)|\\&=\frac{n2^n\pi\mu(\bar{x})}{a}\left|\sin\frac{x-x_0}{2}\right|\cdots\left|\sin\frac{x-x_{2n}}{2}\right|.\end{aligned}$$

Hence assertion (ii) follows. □

Remark 9. Our conjecture is that for any fixed $x'\in[-\pi,\pi]$ the kernel $K(x',t)$ does not change its sign in $[-\pi,\pi]$. If that is true, then the mean value theorem implies the Lagrange-type estimate

$$f(x)-t_n(f,x)=\frac{\widetilde{D}_{n+1}f(\xi_x)}{(n!)^2}\omega(x),\quad x\in[-\pi,\pi],$$

where $f \in C^{2n+1}[-\pi,\pi]$, and

$$\omega(x) = x + a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

is the only function of this form, which vanishes in the nodes $\{x_k\}_{k=0}^{2n}$ and has no other zeroes in $[-\pi,\pi)$. Actually,

$$\omega(x) = x - \sum_{k=0}^{2n} x_k t_{nk}(x).$$

Let the bounded linear operator $L_n : C^*[-\pi,\pi] \to C^*[-\pi,\pi]$ be of the form

$$L_n(f,x) = \mathcal{M}_n * f(x) := \int_{-\pi}^{\pi} \mathcal{M}_n(x-t) f(t)\, dt, \tag{4.6}$$

where $\mathcal{M}_n \in L_1^*[-\pi,\pi]$. For any fixed $t \in [-\pi,\pi]$ we define the 2π-periodic function $\rho_t : \mathbf{R} \to \mathbf{R}$ by

$$\rho_t(x) := 1 - \cos[(x - 2k\pi - t)_+], \quad x \in [(2k-1)\pi, (2k+1)\pi),\ k \in \mathbf{Z}.$$

It is quite easy to verify the following assertion.

Proposition 10. *Let $f \in C^{2n+1}[-\pi,\pi]$ be 2π-periodic. Let also the bounded linear operator L_n, defined by (4.6), preserve the trigonometric polynomials of degree n. Then*

$$f(x) - L_n(f,x) = \frac{1}{n!\,(2n-1)!!} \int_{-\pi}^{\pi} [\rho_t^n(x) - \mathcal{M}_n * \rho_t^n(x)] \widetilde{D}_{n+1} f(t)\, dt.$$

Proof. Making use of formula (1.5) with $c = -\pi$ and changing the order of integration after that, we get easily the estimate

$$\begin{aligned} f(x) - L_n(f,x) &= \frac{1}{n!\,(2n-1)!!} \int_{-\pi}^{\pi} (1 - \cos[(x-t)_+])^n \widetilde{D}_{n+1} f(t)\, dt \\ &- \frac{1}{n!\,(2n-1)!!} \int_{-\pi}^{\pi} \mathcal{M}_n(x-t) \left(\int_{-\pi}^{\pi} (1 - \cos[(t-u)_+])^n \widetilde{D}_{n+1} f(u)\, du \right) dt \\ &= \frac{1}{n!\,(2n-1)!!} \int_{-\pi}^{\pi} \Bigg((1 - \cos[(x-t)_+])^n \\ &- \int_{-\pi}^{\pi} \mathcal{M}_n(x-u)(1 - \cos[(u-t)_+])^n\, du \Bigg) \widetilde{D}_{n+1} f(t)\, dt. \end{aligned}$$

Thus the proof is completed. □

Immediately, Proposition 10 yields

Corollary 11. *Let $f \in C^{2n+1}[-\pi, \pi]$ be 2π-periodic. Let also the bounded linear operator L_n, defined by (4.6), preserve the trigonometric polynomials of degree n. Then*

$$\|f - L_n f\|_\infty \le \frac{2^{n+1}\pi}{n!\,(2n-1)!!}(1 + \|\mathcal{M}_n\|_1)\|\widetilde{D}_{n+1} f\|_\infty.$$

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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EXISTENCE OF STRONG M-BASES IN NONSEPARABLE BANACH SPACES

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In the paper the important question on existence of strong M-bases is considered. A new kind resolution of identity is introduced. Based on this resolution, necessary and sufficient conditions for existence of strong M-bases are determined. As a consequence, the existence of strong M-bases in certain Banach spaces is shown.

Keywords: strong M-basis; resolution of identity; WLD-, WCD-, WCG-space; compact of Valdivia, of Eberlein, of Gulko, of Corson

2000 MSC: 46B26

1. INTRODUCTION

Strong M-bases are natural generalization of Schauder bases in separable Banach spaces. It is known that not every separable Banach space possesses a Schauder basis. In 1994, Terenzi proved that every separable Banach space has a strong M-basis ([10]). The concept of strong M-basis was transfered into nonseparable Banach spaces and its properties were studied by G. Alexandrov ([1]). The existence of strong M-basis in a nonseparable Banach space X leads to more detailed information about the space. For example, it implies the existence of an equivalent local uniformly rotund norm on X ([2]), which has a considerable impact on the geometry and topology of the space. A classical example of a Banach space which possesses a strong M-basis is the space $C[0,\alpha]$ of all continuous functions on the interval $[0,\alpha]$ ([1]).

In this paper we introduce a new kind resolution on a Banach space. Using this resolution, we determine necessary and sufficient conditions for the existence of strong M-bases in nonseparable Banach spaces and apply them to obtain the existence of strong M-bases in certain classes of nonseparable Banach spaces. The question for the existence of strong M-bases is considered also by Deba P. Sinha ([9]). Note that our results are more general and they are announced earlier, on an International Colloquium ([3]).

Let us mention some basic notations used throughout the paper. If α is an ordinal, $|\alpha|$ represents its cardinal number. If A is a set, $|A|$ denotes its cardinal number. ω is the first infinite ordinal. The density character of a topological space X (*dens* X) is defined as the first cardinal number λ such that there is a dense subset A of X with $|A| = \lambda$. If F is a subset of a Banach space X, $linF$ is the linear span of F and $[F]$ denotes the norm-closed linear span of F. Throughout the paper X denotes a Banach space and X^* denotes its dual space. Recall that a *Markushevich basis* (*M-basis*) of X is a biorthogonal system $\{x_i, f_i\}_{i\in I} \subset X \times X^*$ for which $[\{x_i\}_{i\in I}] = X$ and $\{f_i\}_{i\in I}$ is total (i.e. $f_i(x) = 0$ for all $i \in I$ implies $x = 0$). An M-basis of X is said to be a *strong M-basis* of X if

$$\text{every } x \in X \text{ belongs to } [\{f_i(x)x_i\}_{i\in I}]. \tag{1.1}$$

A linear operator $P : X \to X$ on a Banach space X is said to be a projection on X if $P_\alpha^2 = P_\alpha$. The concept Projectional Resolution of Identity (PRI) is well studied and PRI's are constructed on some classes of Banach spaces ([5,6,11,12]). A PRI on X is a collection $\{P_\alpha : \omega \leq \alpha \leq \mu\}$ of continuous projections of X into X, where μ is the smallest ordinal with cardinality $|\mu| = densX$ and for every $\alpha \in [\omega, \mu]$ the following is satisfied:

(i) $P_\alpha P_\beta = P_\beta P_\alpha = P_{min(\alpha,\beta)}$ for every $\beta \in [\omega, \mu]$;

(ii) $P_\mu = Id_X$;

(iii) $densP_\alpha(X) \leq |\alpha|$;

(iv) there exists a constant C such that $\| P_\beta \| \leq C$ for all $\beta \in [\omega, \mu]$;

(v) $\cup\{P_{\beta+1}(X) : \omega \leq \beta < \alpha\}$ is the norm-dense in $P_\alpha(X)$.

Note that the classical concept PRI requires $\|P_\alpha\| = 1$ for all $\alpha \in [\omega, \nu]$, but for the present purpose it is sufficient to have all the projections bounded by the same constant. By [6, p.236], if $\{P_\alpha : \omega \leq \alpha \leq \mu\}$ is a PRI on X with $C = 1$, then

$$\text{every } x \in X \text{ belongs to } [\{P_\omega x\} \cup \{(P_{\beta+1} - P_\beta)x : \omega \leq \beta < \nu\}]. \tag{1.2}$$

It is not difficult to see that (1.2) is valid also for a PRI with $C \neq 1$. Condition (1.2) plays a basic role for the results in the next section. That is why, a new kind resolution based on this condition is introduced there.

At the end of this section we recall the definitions of the spaces used in Section 3. A Banach space X is said to be *Weakly Lindelöf Determined* (WLD) if there exist a set I and a limited linear one-to-one operator $T : X^* \to l_c^\infty(I)$, which is weak-pointwise continuous. A Banach space X is called *Weakly Countably Determined* (WCD) if there exists a countable collection $\{K_n : n \geq 1\}$ of ω^*-compact subsets of X^{**} such that for every $x \in X$ and every $u \in X^{**} \setminus X$ there exists an n_0 such that $x \in K_{n_0}$ and $u \notin K_{n_0}$. A Banach space X is said to be *Weakly Compactly Generated* (WCG) if there exists a weakly compact subset W of X that spans a dense linear subspace in X. For any set $I, \sum(I)$ denotes the subset of $[0,1]^I$ consisting of all functions $\{x(i) : i \in I\}$ such that $x(i) = 0$ except for a countable number of i's. Let K be a compact set. Then K is said to be: *Eberlein compact* if K is homeomorphic to a weakly compact subset of some Banach space X; *Gul'ko compact* if $C(K)$ is weakly countably determined; *Corson compact* if it is homeomorphic to a compact subset of $\sum(I)$ for some I; *Valdivia compact* if there exist a set I and a subset K_0 of $[0,1]^I$ such that K is homeomorphic to K_0 and $K_0 \cap \sum(I)$ is dense in K_0.

2. NECESSARY AND SUFFICIENT CONDITIONS FOR THE EXISTENCE OF STRONG M-BASES

Since condition (1.2) is important for our main theorems, we replace some of the conditions in PRI's definition and consider the following kind of resolution:

Definition 2.1. Let X be a Banach space and ν be an ordinal with cardinality $|\nu| = densX$. A *Semi-projectional Resolution of Identity* (SPRI) on X is a collection $\{P_\alpha : \omega \leq \alpha \leq \nu\}$ of continuous projections of X into X such that:

(i) $P_\alpha P_\beta = P_\beta P_\alpha = P_{min(\alpha,\beta)}$ for every $\alpha, \beta \in [\omega, \nu]$;

(ii) $P_\nu = Id_X$;

(iii) $densP_\alpha(X) < densX$, $\forall \alpha \in [\omega, \nu)$;

(iv) every $x \in X$ belongs to $[\{P_\omega x\} \cup \{(P_{\beta+1} - P_\beta)x : \omega \leq \beta < \nu\}]$.

As was observed above, every PRI on X satisfies (1.2) and hence it is a SPRI on X. One could expect that not every SPRI is a PRI, but a concrete example is not known yet.

The following theorem determines conditions of a resolution on a Banach space X, implying existence of a strong M-basis on X:

Theorem 2.2. *Let ν be an arbitrary ordinal number and let $\{P_\alpha : \omega \leq \alpha < \nu\}$ be a collection of continuous projections of X into X, satisfying the following conditions:*

(i) $P_\alpha P_\beta = P_\beta P_\alpha = P_{min(\alpha,\beta)}$, $\forall \alpha, \beta \in [\omega, \nu)$;

(ii) *each $x \in X$ belongs to $[\{P_\omega x\} \cup \{(P_{\beta+1} - P_\beta)x : \omega \leq \beta < \nu\}]$.*

If there exist strong M-bases of $P_\omega(X)$ and of all $(P_{\alpha+1} - P_\alpha)(X), \alpha \in [\omega, \nu)$, then the space X has a strong M-basis.

Proof. Denote $T_0 = P_\omega$ and $T_\alpha = P_{\alpha+1} - P_\alpha$ for $\alpha \in [\omega, \nu)$. For every $\alpha \in \{0\} \cup [\omega, \nu)$ let $\{x_i^\alpha, f_i^\alpha\}_{i \in I_\alpha}$ be a strong M-basis of $T_\alpha(X)$. For each $\alpha \in \{0\} \cup [\omega, \nu)$ and each $i \in I_\alpha$ define the functional $F_i^\alpha \in X^*$ by the formula $F_i^\alpha(x) = f_i^\alpha(T_\alpha x)$. We will prove that the system $\{x_i^\alpha, F_i^\alpha\}_{\alpha \in \{0\} \cup [\omega, \nu), i \in I_\alpha}$ is a strong M-basis of X. Condition (i) implies that the bounded operators T_α, $\alpha \in \{0\} \cup [\omega, \nu)$, are projections which satisfy

$$T_\alpha T_\beta = \mathcal{O}, \ \forall \alpha \neq \beta, \tag{2.1}$$

where $\mathcal{O}$ is the null operator of X^*. Thus

$$F_i^\alpha(x_j^\beta) = \begin{cases} 1, & \text{if } \alpha = \beta \text{ and } i = j, \\ 0, & \text{otherwise}, \end{cases}$$

which proves the biorthogonality.

Fix now an arbitrary x in X. For every $\alpha \in \{0\} \cup [\omega, \nu)$,

$$T_\alpha x \in \left[\{f_i^\alpha(T_\alpha x)x_i^\alpha\}_{i \in I_\alpha}\right].$$

By condition (ii),

$$x \in \left[\{T_\alpha x\}_{\alpha \in \{0\} \cup [\omega, \nu)}\right]. \tag{2.2}$$

Therefore

$$x \in \left[\{F_i^\alpha(x)x_i^\alpha\}\right]_{\alpha \in \{0\} \cup [\omega, \nu), i \in I_\alpha}. \tag{2.3}$$

It follows from (2.3) that the family $\{F_i^\alpha\}_{\alpha \in \{0\} \cup [\omega, \nu), i \in I_\alpha}$ is total. □

By the result of Terenzi ([10]), asserting that every separable Banach space possesses a strong M-basis, the next corollary is an obvious consequence of Theorem 2.2.

Corollary 2.3. *Let $\{P_\alpha : \omega \leq \alpha < \nu\}$ satisfy the assumptions of Theorem 2.2. If the subspaces $P_\omega(X)$ and $(P_{\alpha+1} - P_\alpha)(X)$, $\alpha \in [\omega, \nu)$, are separable, then there exists a strong M-basis of X.*

Note that for some classes of Banach spaces the existence of a PRI implies the existence of a resolution satisfying the assumptions of the above corollary. Namely, by [6, p. 236], if every element of a given class $\mathcal{P}$ of Banach spaces admits a PRI $\{P_\alpha\}$ such that all $(P_{\alpha+1} - P_\alpha)(X)$ belong to $\mathcal{P}$, then for a given $X \in \mathcal{P}$ with *dens* $X = \mid \mu \mid$ there exists a collection $\{Q_\gamma : \omega \leq \gamma < \mu\}$ of projections of X into X satisfying the SPRI's properties and such that $Q_\omega(X)$ and all $(Q_{\gamma+1} - Q_\gamma)(X)$ are separable. Note that the same assertion can be proved in case the assumption "PRI" is replaced by "SPRI". The next theorem gives sufficient conditions for the existence of a strong M-basis in each element of a given class of nonseparable Banach spaces.

Theorem 2.4. *Let $\mathcal{P}$ be a class of Banach spaces such that for every $X \in \mathcal{P}$ there exists a* SPRI $\{P_\alpha : \omega \le \alpha < \nu\}$ *on X such that $(P_{\alpha+1} - P_\alpha)(X) \in \mathcal{P}$ for every $\alpha \in [\omega, \nu)$. Then each $X \in \mathcal{P}$ has a strong M-basis.*

Proof. We proceed by transfinite induction on the density character of X. If *dens* $X = |\omega|$, i.e. if X is a separable space, then X has a strong M-basis ([10]). Let now *dens* $X > |\omega|$ and let us assume that every space $Z \in \mathcal{P}$ with *dens* $Z <$ *dens* X has a strong M-basis. Let $\{P_\alpha : \omega \le \alpha < \nu\}$ be a SPRI on X such that $(P_{\alpha+1} - P_\alpha)(X) \in \mathcal{P}$ for every $\alpha \in [\omega, \nu)$. Then all the subspaces $P_\omega(X)$ and $(P_{\alpha+1} - P_\alpha)(X), \alpha \in [\omega, \nu)$, have a strong M-basis by the induction hypothesis. Now, applying Theorem 2.2, we obtain that X has a strong M-basis. □

An obvious consequence of the above theorem is the following

Corollary 2.5. *Let $\mathcal{P}$ be a class of Banach spaces such that:*

1) *$\mathcal{P}$ is a hereditary class (i.e. if $X \in \mathcal{P}$ and Y is a subspace of X, then Y also belongs to $\mathcal{P}$);*

2) *each $X \in \mathcal{P}$ admits a SPRI $\{P_\alpha : \omega \le \alpha \le \mu\}$.*

Then each $X \in \mathcal{P}$ has a strong M-basis.

Theorem 2.2 gives sufficient conditions for the existence of strong M-bases. It turns out that properties (2.1) and (2.2) of the bounded projections T_α are also connected with necessary conditions for the existence of strong M-bases:

Theorem 2.6. *A Banach space X has a strong M-basis if and only if there exist a set of ordinals J and a family $\{T_\alpha\}_{\alpha \in J}$ of continuous projections of X into X, which satisfy the following conditions:*

(i) *$T_\alpha T_\beta$ is the null operator on X for every $\alpha \ne \beta$;*

(ii) *every $x \in X$ belongs to the norm-closed linear span of $\{T_\alpha x\}_{\alpha \in J}$;*

(iii) *there exists a strong M-basis in $T_\alpha(X)$ for every $\alpha \in J$.*

Proof. It follows as in the proof of Theorem 2.2 that the existence of bounded projections $\{T_\alpha : X \to X\}_{\alpha \in J}$, satisfying (i)-(iii), implies the existence of a strong M-basis of X. Vice-versa, let $\{x_i, f_i\}_{i \in I}$ be a strong M-basis of X. Since every set can be well ordered [7], order I and let ν be the ordinal number of this order. For every $\alpha \in [0, \nu)$ define the operator $T_\alpha : X \to X$ by $T_\alpha(x) = f_\alpha(x) x_\alpha$, $\forall x \in X$. Then the family $\{T_\alpha\}_{\alpha \in [0,\nu)}$ satisfies conditions (i)-(iii). □

Note that the above theorem remains valid if condition (iii) is replaced by

(iii') *all $T_\alpha(X)$ are separable/finite dimensional.*

It would be interesting to find out whether there exists a Banach space which possesses a strong M-basis and does not possess a PRI. In case such a space exists, it would mean that the resolution used in the above Theorem 2.6 is more proper than PRI when strong M-bases are considered.

3. EXISTENCE OF STRONG M-BASES IN CERTAIN CLASSES OF NONSEPARABLE BANACH SPACES

Since every PRI on a Banach space X is a SPRI on X, Corollary 2.5 remains valid if PRI is used instead of SPRI. Based on this corollary, the existence of strong M-bases in some classes of Banach spaces is obtained.

Proposition 3.1. *If X is either a WLD, a WCD or a WCG-space, then X has a strong M-basis.*

Proof. It is known that the class of all WLD-Banach spaces is hereditary and every WLD-Banach space admits a PRI ([4]). Thus, Corollary 2.5 implies that every WLD-Banach space has a strong M-basis. If X is a WCD or WCG-space, then X is a WLD-space ([4]) and therefore has a strong M-basis. □

Proposition 3.2. *If K is a compact either of Valdivia, of Eberlein, of Gul'ko or of Corson, then there exists a strong M-basis of the space $C(K)$.*

Proof. Let K be a Valdivia compact, $\{P_\alpha : \omega \leq \alpha \leq \mu\}$ be the PRI on $C(K)$, constructed in [6, p.256], and $\mathcal{P}$ be the class of all spaces $C(V)$, where $V's$ are Valdivia compacts. Observe that all subspaces $(P_{\alpha+1} - P_\alpha)(C(K))$ from this construction belong to $\mathcal{P}$. Therefore, by Theorem 2.4, there exists a strong M-basis of $C(K)$. The rest follows trivially, keeping in mind that if K is a compact of Eberlein, Gul'ko or Corson, then K is a compact of Valdivia ([6, p.253]). □

As it is well-known, there exists an orthonormal basis in every separable Hilbert space. Concerning nonseparable Hilbert spaces, let us mention, for example, the space of all almost periodic functions of Bor and the set $\{e^{i\lambda t}\}$, which is a complete orthonormal system for this space ([8]). The next proposition proves the existence of a strong M-basis in every nonseparable Hilbert space.

Proposition 3.3. *Every Hilbert space has a strong M-basis.*

Proof. Let H be a Hilbert space and μ be the smallest ordinal with $|\ \mu\ | = densX$. Fix an arbitrary dense subset $\{x_\beta\}_{1\leq\beta<\mu}$ in H. For every $\alpha \in [\omega, \mu]$ let $L_\alpha = [\{x_\beta\}_{\beta<\alpha}]$ and P_α be the orthogonal projectional operator on L_α. Then the family $\{P_\alpha\}_{\alpha\in[\omega,\mu]}$ is a PRI on H. Finally, apply Corollary 2.5 to the class of all Hilbert spaces. □

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

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ON LOWER BOUNDS OF THE SECOND-ORDER DINI DIRECTIONAL DERIVATIVES

VSEVOLOD I. IVANOV

In this paper we show that the upper Dini directional derivative of an radially upper semicontinuous function has the same lower bounds as the lower Dini directional derivative, and that the second-order upper Dini directional derivative of an radially upper semicontinuous function, which satisfies some additional assumptions, has the same lower bounds as the second-order lower Dini directional derivative. A second-order complete characterization of a convex function is obtained in terms of the second-order upper Dini derivative and of the first-order one. These results are extensions of the respective theorems of L. R. Huang and K. F. Ng.
A second-order Taylor inequality is derived.

Keywords: nonsmooth analysis, lower bounds of Dini directional derivatives, lower bounds of second-order Dini directional derivatives

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1. DINI DERIVATIVES

A lot of derivatives of the nonsmooth functions are introduced mostly for the purpose of optimization. The Dini derivatives play a key role among them.

In the sequel $\mathbf{E}$ is a real normed vector space, the real finite-valued function f is defined on the open set $X \subset \mathbf{E}$. The set of reals is denoted by $\mathbb{R}$, and $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$. Consider the following generalized directional derivatives of f at the point $x \in X$ in the direction $u \in \mathbf{E}$:

$$f'_+(x; u) = \limsup_{t \downarrow 0} t^{-1}(f(x + tu) - f(x)),$$

$$f'_-(x; u) = \liminf_{t \downarrow 0} t^{-1}(f(x + tu) - f(x)).$$

They are usually called respectively upper and lower Dini directional derivatives.

The following theorem claims that the upper Dini directional derivative of an radially upper semicontinuous (radially u.s.c. for short) function has the same lower bounds as the lower one.

Theorem 1. *Let $X \subset \mathbf{E}$ be an open convex set, and $f : X \to \mathbb{R}$ be a radially u.s.c. function. Suppose that $u \in \mathbf{E}$, and $\alpha \in \overline{\mathbb{R}}$. Then the following implications hold:*

$$f'_+(x; u) \geq \alpha, \ \forall\, x \in X \quad \Longleftrightarrow \quad f'_-(x; u) \geq \alpha, \ \forall\, x \in X, \tag{1.1}$$

$$f'_+(x; u) \geq \alpha, \ \forall\, x \in X \quad \Longrightarrow \quad f(x + tu) - f(x) - \alpha t \geq 0, \tag{1.2}$$

$$\forall\, x \in X, \ \forall\, t \geq 0 \quad \textit{provided that} \quad x + tu \in X.$$

Proof. Assume that $f'_+(x; u) \geq \alpha$ for all $x \in X$. If $\alpha = -\infty$, then the claim is obvious. Let $\alpha > -\infty$, and β be an arbitrary number such that $\beta < \alpha$. Suppose that $x \in X$ is fixed. There exists a sequence t_n of real positive numbers, converging to 0, such that

$$t_n^{-1}(f(x + t_n u) - f(x)) > \beta. \tag{1.3}$$

Consider the function

$$\psi(t) = f(x + tu) - f(x) - \beta t,$$

which is defined for all $t \geq 0$ such that $x + tu \in X$, and the set

$$A = \{t \in (0, \infty) \mid x + tu \in X, \ \psi(t) > 0\}.$$

It is clear that $t_n \in A$, and $\inf A = 0$. We show that A is an interval with the right endpoint

$$b = \sup\{t \in (0, \infty) \mid x + tu \in X\}.$$

Indeed, suppose that there exists $c \in \mathbb{R}$, satisfying $0 < c < b$, $\psi(c) \leq 0$. Since ψ is u.s.c., then by the generalized Weierstrass theorem there exists a global maximizer ξ of ψ over the closed interval $[0, c]$. It follows from (1.3) that there exists $t \in A$ such that $0 < t < c$. Hence $\psi(\xi) \geq \psi(t) > 0$, and $0 < \xi < c$. According to the necessary maximality condition, $\psi'_+(\xi; 1) \leq 0$. On the other hand,

$$\psi'_+(\xi; 1) = f'_+(x + \xi u; u) - \beta \geq \alpha - \beta > 0,$$

which is a contradiction. Consequently, b is the right endpoint of A, and A is an interval. For all sufficiently small $t > 0$ we have $t^{-1}(f(x + tu) - f(x)) > \beta$. Therefore $f'_-(x; u) \geq \beta$. Since β is arbitrary such that $\beta < \alpha$, then $f'_-(x; u) \geq \alpha$.

The converse implication of (1.1) is obvious.

We shall prove the inequality (1.2). Since b does not depend of β, we have $t^{-1}(f(x+tu)-f(x)) > \beta$ for all β and t such that $\beta < \alpha$, $t > 0$, $x+tu \in X$. Since β is arbitrary, then (1.2) holds. □

Example 1. The following example shows that the assumption f to be radially u.s.c. cannot be dropped in Theorem 1. Consider the function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = \begin{cases} 0, & x \text{ is rational,} \\ 1, & \text{otherwise.} \end{cases}$$

The number 0 is a lower bound of the upper Dini derivative, since $f'_+(x;1) \geq 0$ for all $x \in \mathbb{R}$. If x is irrational, then $f'_-(x;1) = -\infty$.

Example 2. The following example shows that the assumption f to be radially u.s.c. cannot be dropped in Theorem 1 even in the case when the function is lower semicontinuous (l.s.c.). Consider the function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = \begin{cases} 0, & x \text{ is irrational,} \\ -1/q, & x = (p/q),\ p,\ q \text{ are integers.} \end{cases}$$

The number 0 is a lower bound of the upper Dini derivative. The number $\sqrt{2}$ is an endless decimal $1,4142...$ The sequences $1,\ 10,\ 10^2,\ 10^3,\ 10^4, \ldots$ and $1+2,\ 14+2,\ 141+2,\ 1414+2,\ 14142+2, \ldots$ correspond to the value of this number. Denote them respectively by q_n and p_n. It is obvious that

$$f'_-(\sqrt{2};1) \leq \liminf_{\frac{p}{q} \to \sqrt{2},\ \frac{p}{q} > \sqrt{2},\ p,\ q\ -\ integers} \frac{-1/q}{(p/q)-\sqrt{2}} \leq \liminf_{n\to\infty} \frac{-1}{p_n - q_n\sqrt{2}}.$$

Since $-1 < p_n - 2 - q_n\sqrt{2} < 0$, then $-1/(p_n - q_n\sqrt{2}) < -1/2$ and $f'_-(\sqrt{2};1) \leq -1/2$.

Now we show some applications of Theorem 1.

The following result is a direct consequence of the Zygmund's lemma (see, for example, Penot [7, Lemma 1.1]). Some of its proofs can be found in Diewert [3, Corollary 4 and 5], Giorgi and Komlosi [5, Theorem 1.13] and references therein. See Scheffler [9, Lemma 4.1], too.

Corollary 1. *Let $\varphi : [a,b] \to \mathbb{R}$ be an u.s.c. function. If*

$$\varphi'_+(x;1) \geq 0 \ (\varphi'_+(x;1) > 0) \text{ for all } x \in [a,b),$$

then φ is monotone nondecreasing (strictly monotone increasing) on $[a,b)$.

Proof. Let $\varphi'_+(x;1) \geq 0$ for all $x \in [a,b)$, and $a \leq x_1 < x_2 < b$. Choosing $\alpha = 0$, it follows from (1.2) that $\varphi(x_2) \geq \varphi(x_1)$, since the function φ can be continued in a constant manner to the left of the point a to obtain an open interval, where the right upper Dini derivative is nonnegative.

Assume that $\varphi'_+(x;1) > 0$ for all $x \in [a,b)$. Using the arguments of Theorem 1, by choosing $\beta = 0$, we get that (1.2) will be strict with $\alpha = 0$, i.e. φ is strictly monotone increasing. □

The following statement is well known (see, for example, Giorgi and Komlosi [5, Theorem 1.10]), but our proof is shorter.

Corollary 2. *Let $\varphi : [a,b] \to \mathbb{R}$ be an u.s.c. function. If the function*

$$h(t) = \varphi(t) - \varphi(a) - \gamma(t-a), \quad \textit{where} \quad \gamma = \frac{\varphi(b) - \varphi(a)}{b-a},$$

assumes a global minimum over $[a,b]$, then there exists an intermediate point t_1 such that $\varphi'_+(t_1;1) \leq \gamma$.

Proof. Suppose the contrary that $\varphi'_+(t;1) > \gamma$ for all $t \in (a,b)$. Therefore, $h'_+(t;1) = \varphi'_+(t;1) - \gamma > 0$ for all $t \in (a,b)$. According to Corollary 1, h is strictly monotone increasing on (a,b). The function h is u.s.c. Since $h(a) = h(b) = 0$, by the upper semicontinuity, $h(t) < 0$ when $t \in (a,b)$. Then h cannot assume a minimal value over $[a,b]$, which is a contradiction. □

The following is a well known version of the mean value theorem. Similar results are proved in Demyanov and Rubinov [2, Theorem 1.3.1], Giorgi and Komlosi [5, Corollary 1.9], Penot [7, Proposition 1.3] and references therein.

Corollary 3. *Let $\varphi : [a,b] \to \mathbb{R}$ be an u.s.c. function. Then*

$$\varphi(b) - \varphi(a) \geq m(b-a), \quad \textit{where} \quad m = \inf_{a \leq z < b} \varphi'_+(z;1).$$

Proof. Denote $g(t) = \varphi(a+t) - \varphi(a) - mt$. It is defined and u.s.c. for all $t \in [0, b-a]$. Since $g'_+(t;1) = \varphi'_+(a+t;1) - m \geq 0$ for all $t \in [0, b-a)$, by Corollary 1, g is monotone nondecreasing. Therefore, $g(t) \geq g(0)$ for all $t \in [0, b-a)$. Since g is u.s.c., then $g(b-a) \geq \limsup_{t \to b-a} g(t) \geq g(0)$. Hence, $\varphi(b) - \varphi(a) - m(b-a) \geq 0$. □

2. SECOND-ORDER DINI DERIVATIVES

There are several ways to define second-order Dini derivatives. One of them is the following. Consider the function $f : X \to \mathbb{R}$, where $X \subset \mathbf{E}$ is an open set. We define the second-order upper Dini derivative of f at $x \in X$ in the direction $u \in \mathbf{E}$ and the lower one as follows:

$$f''_+(x;u) = \limsup_{t \downarrow 0} 2t^{-2}(f(x+tu) - f(x) - tf'_+(x;u)),$$

$$f''_-(x;u) = \liminf_{t\downarrow 0} 2t^{-2}(f(x+tu) - f(x) - tf'_-(x;u)).$$

We call f''_+ upper and f''_- lower in consistence with the first-order derivatives, but Example 1 shows that the inequality $f''_-(x;u) \leq f''_+(x;u)$ may be violated for some $x \in X$, $u \in \mathbf{E}$. If x is rational, then $f'_+(x;1) = \infty$, $f'_-(x;1) = 0$, $f''_+(x;1) = -\infty$, $f''_-(x;1) = 0$.

The following theorem is connected to Theorem 1.

Theorem 2. *Let $X \subset \mathbf{E}$ be an open convex set, and $f : X \to \mathbb{R}$ be a radially u.s.c. function. Suppose that $u \in \mathbf{E}$, $\alpha \in \overline{\mathbb{R}}$, and $f'_+(x;u) + f'_+(x;-u) \geq 0$ for all $x \in X$. Then the following implications hold:*

$$f''_+(x;u) \geq \alpha,\ \forall\, x \in X \implies f''_-(x;u) \geq \alpha,\ \forall\, x \in X; \tag{2.1}$$

$$f''_+(x;u) \geq \alpha,\ \forall\, x \in X \implies f(x+tu) - f(x) - tf'_+(x;u) \geq 0.5t^2\alpha,$$
$$\forall\, x \in X,\ \forall\, t > 0 \text{ such that } x + tu \in X.$$

When f is directional differentiable everywhere, i.e.

$$f'_+(x;u) = f'_-(x;u) = f'(x;u),\ \forall\, x \in X,\ \forall\, u \in \mathbf{E},$$

then the converse implication of (2.1) *holds.*

Proof. The case when $\alpha = -\infty$ is evident. Assume that $\alpha > -\infty$ and $f''_+(x;u) \geq \alpha$ for all $x \in X$. For arbitrary fixed $x \in X$ and $\beta \in \mathbb{R}$, satisfying $\beta < \alpha$, consider the function

$$\psi(t) = f(x+tu) - f(x) - tf'_+(x;u) - 0.5\beta t^2,$$

which is defined for all $t \geq 0$ such that $x + tu \in X$, and the set

$$A = \{t \in (0,\infty) \mid x + tu \in X,\ \psi(t) > 0\}.$$

Then $A \equiv (0,b)$, where $b = \sup\{t \in (0,\infty) \mid x + tu \in X\}$. Indeed, it follows from $f''_+(x;u) > \beta$ that there exists a sequence of positive numbers t_n converging to 0, which satisfy the inequality $\psi(t_n) > 0$. Hence $\inf A = 0$. Let there exist $c \in \mathbb{R}$ such that $0 < c < b$ and $\psi(c) \leq 0$. According to the inequality $\psi(t_n) > 0$, there exists $t \in (0,c) \cap A$. By the upper semicontinuity of ψ, there exists a global maximizer ξ of ψ over $[0,c]$. Since $\psi(\xi) \geq \psi(t) > 0$, then $0 < \xi < c$. On the other hand, we have

$$\psi'_+(\xi;1) = f'_+(x+\xi u;u) - f'_+(x;u) - \beta\xi,$$

$$\psi'_+(\xi;-1) = f'_+(x+\xi u;-u) + f'_+(x;u) + \beta\xi.$$

We conclude from the necessary maximality condition that $\psi'_+(\xi;v) \leq 0$ when $v = \pm 1$. Using the hypothesis of the theorem, we get

$$0 \geq \psi'_+(\xi;1) + \psi'_+(\xi;-1) = f'_+(x+\xi u;u) + f'_+(x+\xi u;-u) \geq 0.$$

Therefore $\psi'_+(\xi;1) = 0$. According to the second-order necessary maximality condition, $\psi''_+(\xi;1) \leq 0$. We continue

$$\psi''_+(\xi;1) = f''_+(x+\xi u;u) - \beta \geq \alpha - \beta > 0,$$

which is a contradiction. Consequently, $A \equiv (0,b)$. Since b does not depend of β,

$$2t^{-2}(f(x+tu) - f(x) - tf'_+(x;u)) > \beta$$

for all $t \in (0,b)$ and for arbitrary $\beta < \alpha$. Thus,

$$f(x+tu) - f(x) - tf'_+(x;u) \geq 0.5\alpha t^2,$$

and

$$2t^{-2}(f(x+tu) - f(x) - tf'_-(x;u)) \geq 2t^{-2}(f(x+tu) - f(x) - tf'_+(x;u)) \geq \alpha$$

for all $t \in (0,b)$. Then taking the limits, as $t \to 0$, we get that $f''_-(x;u) \geq \alpha$.

In case when f is directional differentiable everywhere, one easily gets the converse claim to (2.1), since

$$f''_+(x;u) \geq f''_-(x;u),\ \forall\, x \in X,\ \forall\, u \in \mathbf{E}.\ \square$$

The following theorem is a necessary and sufficient condition for convexity.

Theorem 3. *Let $X \subset \mathbf{E}$ be an open convex set, $f : X \to \mathbb{R}$ be a radially u.s.c. function. Then f is convex iff the following conditions hold together:*

$$f'_+(x;u) + f'_+(x;-u) \geq 0 \quad \textit{for all } x \in X,\ u \in \mathbf{E}, \tag{2.2}$$

$$f''_+(x;u) \geq 0 \quad \textit{for all } x \in X,\ u \in \mathbf{E}. \tag{2.3}$$

If inequalities (2.2),(2.3) *hold, and* (2.3) *is strict for all $x \in X$, $u \in \mathbf{E}$, then f is strictly convex.*

Proof. It is obvious that each convex function satisfies inequalities (2.2), (2.3). Conversely, suppose that (2.2), (2.3) are fulfilled. Applying Theorem 2 by choosing $\alpha = 0$, we obtain that

$$f(x+tu) - f(x) \geq tf'_+(x;u) \text{ for all } t \text{ such that } 0 \leq t < b, \tag{2.4}$$

where $b = \sup\{t \in (0,\infty) \mid x + tu \in X\}$. It follows from (2.4) that for all $x' \in X$, $y' \in X$, $\lambda \in [0,1]$ the following inequalities are fulfilled:

$$f(x') - f(x' + \lambda(y'-x')) \geq \lambda f'_+(x' + \lambda(y'-x'); x'-y'), \tag{2.5}$$

$$f(y') - f(x' + \lambda(y'-x')) \geq (1-\lambda) f'_+(x' + \lambda(y'-x'); y'-x'). \tag{2.6}$$

By using (2.2), we infer from (2.5) and (2.6) that

$$(1-\lambda)f(x') + \lambda f(y') - f(x' + \lambda(y' - x')) \geq$$
$$\lambda(1-\lambda)(f'_{+}(x' + \lambda(y' - x'); x' - y') + f'_{+}(x' + \lambda(y' - x'); y' - x')) \geq 0.$$
Therefore f is convex.

The *strictly convex case is similar. We must take only* $\beta = 0$. Then it is seen from Theorem 2 that inequality (2.4) will be strict. □

Theorems 2 and 3 are extensions of Theorems 1 and 2 in Huang and Ng [6], where they are proved in the case when the function is locally Lipschitz and regular in the sense of Clarke [1]. But inequality (2.2) is not used in Theorem 2 of Huang and Ng [6]. A locally Lipschitz regular function always fulfills it.

Remark 1. For example, some classes of functions, which satisfy inequality (2.2), are the Gâteaux-differentiable, quasidifferentiable in the sense of Pschenichnyi [8], or locally-Lipschitz regular in the sense of Clarke [1] functions. Another functions, which fulfill this condition, are all ones such that the upper Dini subdifferential
$$\partial f(x) := \{\xi \in \mathbf{E}^* \mid \langle \xi, u \rangle \leq f'_{+}(x; u) \; \forall\, u \in \mathbf{E}\}$$
is nonempty for all $x \in X$. The functions of the first three classes from above are directional differentiable.

The following is an application of Theorem 3, and it says when a second-order Taylor inequality holds.

Theorem 4. *Let* $\varphi : [a, b] \to \mathbb{R}$ *be an u.s.c. function. Assume that*
$$\varphi'_{+}(x; 1) + \varphi'_{+}(x; -1) \geq 0, \qquad \forall\, x \in (a, b).$$
Then $\varphi(b) - \varphi(a) - (b - a)\varphi'_{+}(a; 1) \geq 0.5m(b - a)^2$, *where*
$$m = \min\{\inf_{a<x<b} \varphi''_{+}(x; 1), \inf_{a<x<b} \varphi''_{+}(x; -1)\}.$$

Proof. Consider the function
$$g(t) = \varphi(a + t) - \varphi(a) - t\varphi'_{+}(a; 1) - 0.5mt^2, \; t \in [0, b - a].$$
It is clear that for all $t \in (0, b - a)$
$$g'_{+}(t; 1) = \varphi'_{+}(a + t; 1) - \varphi'_{+}(a; 1) - mt,$$
$$g'_{+}(t; -1) = \varphi'_{+}(a + t; -1) + \varphi'_{+}(a; 1) + mt.$$
Therefore,
$$g'_{+}(t; 1) + g'_{+}(t; -1) = \varphi'_{+}(a + t; 1) + \varphi'_{+}(a + t; -1) \geq 0.$$
Since
$$g''_{+}(t; 1) = \varphi''_{+}(a + t; 1) - m \geq 0,$$

$$g''_+(t;-1) = \varphi''_+(a+t;-1) - m \geq 0,$$

then, by Theorem 3, g is a convex function on $(0, b-a)$. Hence, there exists the directional derivative $g'(t;1) = g'_+(t;1)$ for all $t \in (0, b-a)$. It is easy to verify that there exists $g'_+(0;1)$ and it is equal to 0. Using the upper semicontinuity, it is easy to show that g is convex on $[0, b-a]$.

Suppose that $0 < t < s < s+t < b-a$. By convexity of g, the following inequalities hold:

$$g(s) \leq \frac{t}{s}g(t) + (1-\frac{t}{s})g(s+t),$$

$$g(t) \leq \frac{t}{s}g(s) + (1-\frac{t}{s})g(0).$$

Consequently, $t^{-1}(g(t)-g(0)) \leq t^{-1}(g(s+t)-g(s))$. Taking the limits as $t \to 0$, we get that $0 = g'_+(0;1) \leq g'_+(s;1) = g'(s;1)$ for all $s \in (0, b-a)$. By Corollary 1, g is monotone nondecreasing on $[0, b-a)$. Using the upper semicontinuity, we get

$$g(b-a) \geq \limsup_{s \to b-a,\, s<b-a} g(s) \geq g(0) = 0,$$

which completes the proof. □

Similar, but different results to Theorems 3, 4 are derived by Ginchev and the author [4] in terms of other lower derivatives.

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LETTER TO THE EDITOR

TURÁN'S THEOREM AND MAXIMAL DEGREES

NIKOLAY KHADZHIIVANOV and NEDYALKO NENOV

The number of the edges of a graph G will be denoted by $e(G)$, and the subgraph induced by the neighbours of the vertex x – by G_x. For the n-vertex r-partite Turán's graph $T_r(n)$ we define $e(T_r(n)) = t_r(n)$.

B. Bollobàs in [1] and [2] considered the following simple algorithm to construct a large clique in a graph G: "Pick a vertex x_1 of maximal degree in $G_1 = G$, then a vertex x_2 of maximal degree in $G_2 = G_{x_1}$, and so on. The algorithm stops with x_l if x_l has no neighbours in G_l". In [1] this algorithm was called the degree-greedy algorithm.

B. Bollobas in [1] proved the following results:

Theorem 2 (see also Theorem 5 in [2]). *Let G be a graph with n vertices and $t_r(n)+a$ edges, where $a \geq 0$. Let x be a vertex of maximal degree d. Then $e(G_x) \geq t_{r-1}(d) + a$, and the inequality is strict unless $n - d = \left\lfloor \frac{n}{r} \right\rfloor$, and $G = G_x + \overline{K}_{n-d}$.*

Theorem 5 (see also Theorem 6 in [2]). *Let G be a graph with n vertices and least $t_r(n)$ edges. Then either $G = T_r(n)$ or else the degree-greedy algorithm constructs a clique of G of order at last $r+1$.*

The aim of this note is to draw attention to the fact that:

1. Theorem 2 is a special case of Proposition 1, p. 235 in [3] ($\langle A \rangle = G_x$ for some vertex x of maximal degree), because the case $a = 0$ of Theorem 2 is obviously equivalent to Theorem 2.

2. The degree-greedy algorithm found by B. Bollobas, A. Thomason and J. Bondy in 1983 is a fact published in [4], p. 119. by N. Khadzhiivanov and N. Nenov in 1976.

3. Theorem 5 coincides with Corollary 1 from [4], p. 121.

4. Theorem 3 in [1] is not correct, because $\triangle(T_r(n)) \neq (r-k-1)s+p$.

5. In the proof of Theorem 5, given in [1] and [2], equalities $G_x = T_{r-2}(n-k)$ and $G = T_{r-1}(n)$ are not correct.

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